Review of cell-complexes

Construction.

(1) Start with a discrete set X^0 whose points are 0-cells.

(2) Inductively build X^n from X^{n-1} by attaching n-cells e^n_α via maps $\varphi_\alpha : S^{n-1} \longrightarrow X^{n-1}$.

As a quotient space,

$$X^n = X^{n-1} \sqcup_\alpha D^n_\alpha / x \sim \varphi_\alpha(x),$$

$\forall \quad x \in \partial D^n_\alpha$.

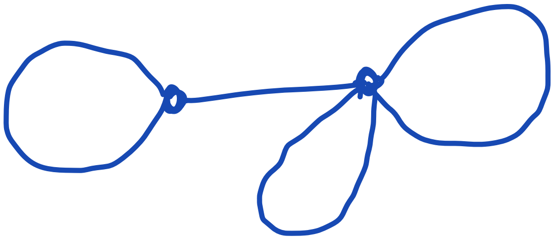
(3) If the process stops at a finite stage, then $X = X^n$, for $n < \infty$.

Otherwise, it can continue indefinitely, setting $X = \bigcup_n X^n$.

A space X constructed in this manner is called a cell-complex (or cw-complex).

Examples

(a) A 1-dimensional cell-complex is called a graph.



(b) $S^n = 0\text{-cell} \sqcup n\text{-cell}$, where the n-cell is attached by a constant map $S^{n-1} \longrightarrow e^0$.

(c) $\mathbb{RP}^n = S^n/x\sim -x$

$= D^n/x\sim -x$, for $x \in \partial D^n$
$= S^{n-1}$

Since $S^{n-1}/x\sim -x = \mathbb{RP}^{n-1}$, we have $\mathbb{RP}^n = \mathbb{RP}^{n-1} \sqcup e^n$ with the quotient projection:

$$S^{n-1} \longrightarrow \mathbb{RP}^{n-1}.$$

$$\mathbb{RP}^\infty = \bigcup_n \mathbb{RP}^n.$$

A <u>subcomplex</u> is a closed subspace $A \subset X$ that is a union of cells of X.

<u>Example</u>

(a) S^n is a subcomplex of S^{n+1} by regarding

S^n as the equator of S^{n+1} and then attaching 2 - $(n+1)$ cells via the identity map $\partial D^{n+1} (= S^n) \longrightarrow S^n$.

$$S^\infty = \bigcup_n S^n.$$

Homology

Why do we need homology?

(a) $\pi_1(X)$ = Homotopy classes of maps $S^1 \longrightarrow X$ can (i.e. homotopy classes of loops based at a point) can only be used to study objects of dimension up to 2. For example, it cannot be used to distinguish between S^n, for $n \geqslant 2$.

(b) The higher-dimensional analogue $\pi_n(X)$ – homotopy classes of maps $S^n \longrightarrow X$ is incredibly hard to compute in general.

It is known that $\pi_i(S^n) = 0$ for $i < n$ and $\mathbb{Z}$ for $i = n$.

(c) The homology groups $H_i(X)$ are easier to compute and depend only on the $(n+1)$-skeleton

Also,

$$H_i(S^n) \cong \begin{cases} \pi_i(S^n), & \text{for } i \leq n \\ 0, & \text{for } i > n. \end{cases}$$

Simplical homology

Defn. An n-simplex is the smallest convex set in $\mathbb{R}^m$ containing n+1 points $v_0, \ldots, v_n$ that do not lie in a hyperplane of dimension $< n$.

(Equivalently, $v_1 - v_0, \ldots, v_n - v_0$ are linearly independent)

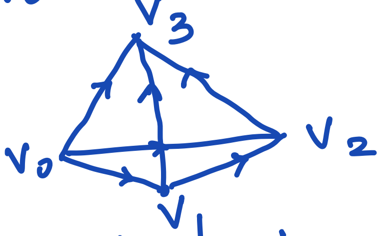
The points v_i are the <u>vertices</u> of the simplex and the simplex will be denoted by $[v_0, \dots, v_n]$.

Examples

(a) 0-simplex :

(b) 1-simplex :

(c) 2-simplex :



Tetrahedron

(d) A standard n-simplex in $\mathbb{R}^m$ will be denoted by :

$$\Delta^n = \left\{ (t_0, \dots, t_n) \in \mathbb{R}^{n+1} \;\middle|\; \sum_i t_i = 1 \text{ and } t_i \geqslant 0 \ \forall i \right\}$$

Remark. (a) For the purposes of homology, it is essential that there is an ordering of the vertices.

(b) This, in turn, induces an orientation on the edges.

(c) There is an induced canonical homeomorphism from the standard simple Δ^n onto any other n-simplex $[v_0, \dots, v_n]$ preserving order of vertices, namely:

$$(t_0, \dots, t_n) \longmapsto \sum_i t_i v_i$$

The t_i are called the barycentric coordinates of the point $\sum t_i v_i$ in $[v_0, \dots, v_n]$.

__Defn__. A __face__ of a simplex is the subsimplex with vertices any nonempty subset of the v_i's.

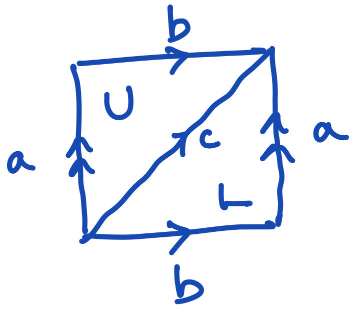
__Remark__. Vertices of a face will always be ordered according to the order of the larger simplex.

__Defn__. A Δ-complex is the quotient space of a collection of disjoint simplices Δ_α^n of various dimensions obtained by identifying certain of their faces via canonical linear homeos that preserve the ordering of vertices.

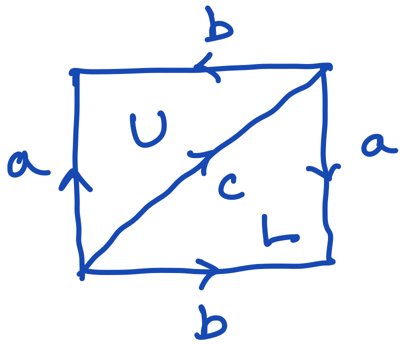
Remark. The data determining a Δ-complex is purely combinatorial (i.e. building something from a kit of precut parts that come together following instructions).

Example.

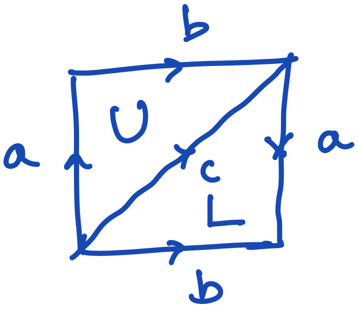
(a) T: Torus



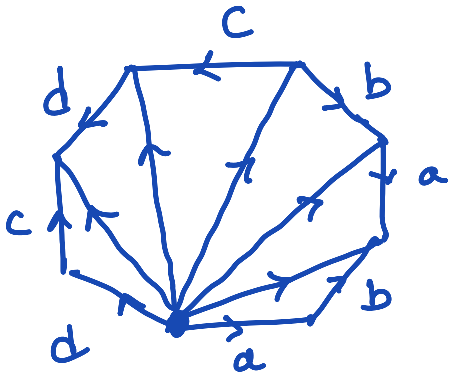
(b) $\mathbb{RP}^2$



(c) Klein-bottle



(d)



S_2.

Remark

(a) As the orientation of the various edges in the boundary of each n-simplex is related to the $[v_0, \ldots, v_n]$, no 2-simplex has its edges oriented cyclically.

(b) Since identification preserves orientation, no two points in the interior are identified.

Defn. We define $\Delta_n(X)$ be the free abelian group with basis the open n-simplices e^n_α of X.

In other words,

$$\Delta_n(x) \cong \bigoplus_{i=1}^{k_n} \mathbb{Z}, \text{ where}$$

k_n - number of n-simplices in X.

Remark. Note that the elements of $\Delta_n(x)$ are finite formal sums $\sum_{\alpha} n_\alpha e_\alpha^n$ called **n-chains**.

Defn. We define boundary $\partial_n(\sigma)$ of an n-simplex $\sigma = [v_0, \dots, v_n]$ to be:

$$\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i \sigma|_{[v_0,\dots,\hat{v}_i \dots, v_n]},$$

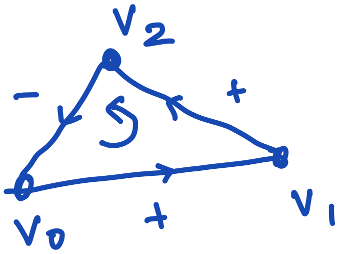
where ^ over v_i indicates the deletion of the vertex v_i.

Example.

(a) 1-simplex :

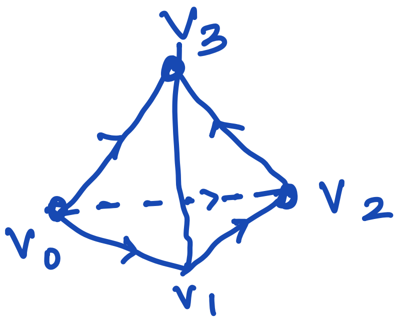
$$\partial_1([v_0, v_1]) = v_1 - v_0$$

(b) 2-simplex :



$$\partial_2([v_0, v_1, v_2]) = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$$

(c) 3-simplex



$$\partial_3([v_0, v_1, v_2, v_3]) = [v_1, v_2, v_3] - [v_0, v_2, v_3] + [v_0, v_1, v_3] - [v_0, v_1, v_2].$$

Defn. The notion of boundary of an n-simplex generalizes to a boundary homomorphism $\partial_n : \Delta_n(X) \longrightarrow \Delta_{n-1}(X)$ on n-chains defined as follows.

Given $\sigma = \sum_\alpha n_\alpha \sigma_\alpha \in \Delta_n(X)$, where $\sigma_\alpha = [v_0^\alpha, \dots, v_n^\alpha]$, we have:

$$\partial_n(\sigma) = \sum_\alpha n_\alpha \sum_{i=0}^{n} (-1)^i \sigma_\alpha \big|[v_0^\alpha, \dots, \hat{v}_i^\alpha, \dots v_n^\alpha].$$

Remark. Note that ∂_n is indeed a homomorphism. (Check!)

Lemma. The composition

$$\Delta_n(X) \xrightarrow{\partial_n} \Delta_{n-1}(X) \xrightarrow{\partial_{n-1}} \Delta_{n-2}(X)$$

is zero.

Proof. It suffices to show for the n-simplices. For an n-simplex $\sigma=[v_0,\dots,v_n]$,

$$\partial_n(\sigma) = \sum_i (-1)^i \sigma|[v_0,\dots,\hat{v_i},\dots v_n]$$

Then:

$$\begin{aligned}\partial_{n-1}(\partial_n(\sigma)) &= \sum_{j<i}(-1)^i(-1)^j \sigma|[v_0,\dots,\hat{v_j},\dots,\hat{v_i},\dots,v_n] \\ &+ \sum_{j>i}(-1)^i(-1)^{j-1} \sigma|[v_0,\dots,\hat{v_i},\dots,\hat{v_j},\dots,v_n] \\ &= 0.\end{aligned}$$

∎

Remark. An immediate consequence of the lemma is:

$$\mathrm{Im}(\partial_n) < \mathrm{Ker}(\partial_{n-1})$$

Thus, we have a sequence

$$\cdots \to C_{n+1}(X) \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \to \cdots$$
$$\cdots \longrightarrow C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} 0$$

of abelian groups with

$\partial_n \partial_{n+1} = 0 \quad \forall n.$

Such a sequence is called a <u>chain complex</u>.

<u>Define</u>. We define the <u>nth simplicial homology group</u> $H_n^{\Delta}(X)$ of X by:

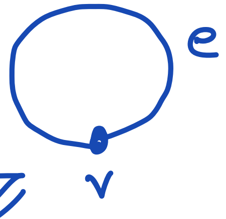
$$H_n^{\Delta}(X) = \mathrm{Ker}(\partial_n) / \mathrm{Im}(\partial_{n+1})$$

where $C_n(X) = \Delta_n(X), \ \forall n.$

The elements of $\operatorname{Ker}(\partial_n)$ are called <u>cycles</u> and the elements of $\operatorname{Im}(\partial_{n+1})$ are called boundaries. Then elements of $H_n(X)$ are called <u>homology classes</u>.

<u>Examples</u>

(a) $X = S^1$



$$\Delta_0(S^1) = \Delta^1(S^1) = \mathbb{Z}$$

$$\begin{tikzcd}
 & \langle e\rangle \arrow[d, phantom, "\parallel"] & & \langle v\rangle \arrow[d, phantom, "\parallel"] & \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "\partial_1"] \arrow[d, phantom, "\parallel"] & \mathbb{Z} \arrow[r, "\partial_0"] \arrow[d, phantom, "\parallel"] & 0 \\
 & C_1(x) & C_0(x) &
\end{tikzcd}$$

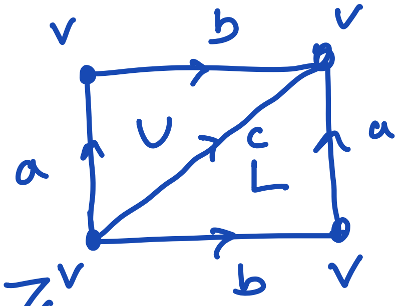
$$\partial_1(e) = v - v = 0 \Rightarrow \partial_1 = 0$$

$$\partial_0 = 0$$

$$H_0^{\Delta}(X) = \frac{\mathrm{Ker}(\partial_0)}{\mathrm{Im}(\partial_1)} = \frac{\mathbb{Z}}{\{0\}} \cong \mathbb{Z}$$

$$H_1^{\Delta}(X) = \frac{\mathrm{Ker}(\partial_1)}{\mathrm{Im}(\partial_2)} = \frac{\mathbb{Z}}{\{0\}} \cong \mathbb{Z}$$

(b) $X = T$



$\Delta_0(X) = \langle v \rangle \cong \mathbb{Z}$

$\Delta_1(X) = \langle a, b \rangle \cong \mathbb{Z} \oplus \mathbb{Z}$

$\Delta_2(X) = \langle U, L \rangle \cong \mathbb{Z} \oplus \mathbb{Z}$

Chain complex

$$\cdots \longrightarrow 0 \longrightarrow \overset{\langle U,L\rangle}{\overset{\shortparallel}{\mathbb{Z}\oplus\mathbb{Z}}} \xrightarrow{\partial_2} \overset{\langle a,b,c\rangle}{\overset{\shortparallel}{\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}}} \xrightarrow{\partial_1} \overset{\langle v\rangle}{\overset{\shortparallel}{\mathbb{Z}}} \xrightarrow{\partial_0} 0$$

$\partial_0 = 0$

$\partial_1(a) = \partial_1(b) = \partial_1(c) = 0 \Longrightarrow \partial_1 = 0$

$\partial_2(U) = a + b - c, \ \partial_2(L) = b + a - c$

$$H_0^\Delta(X) = \frac{\operatorname{Ker}(\partial_0)}{\operatorname{Im}(\partial_1)} = \frac{\mathbb{Z}}{\{0\}} = \mathbb{Z}$$

$$H_1^\Delta(X) = \frac{\operatorname{Ker}(\partial_1)}{\operatorname{Im}(\partial_2)} = \frac{\langle a,b,c\rangle}{\langle a+b-c\rangle}$$

$$= \frac{\langle a,b,a+b-c\rangle}{\langle a+b-c\rangle}$$

$$\cong \mathbb{Z}^2$$

$$H_2^\Delta(x) = \frac{\operatorname{Ker}(\partial_2)}{\operatorname{Im}(\partial_3)}$$

$$\operatorname{Ker}(\partial_2) = \{pU+qL \mid \partial_2(pU+qL)=0\}$$

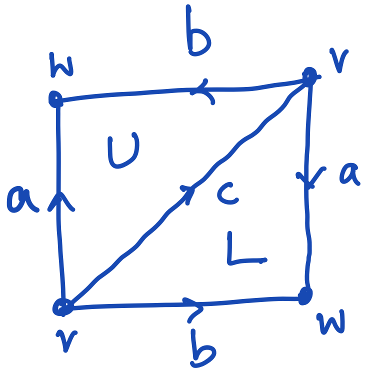
$$= \{pU+qL \mid (p+q)(a+b-c)=0\}$$

$$= \{pU+qL \mid p=-q\}$$

$$= \langle U-L\rangle$$

$$\Rightarrow \frac{\mathrm{Ker}(\partial_2)}{\mathrm{Im}(\partial_3)} = \frac{\langle U-L\rangle}{\{0\}} \cong \mathbb{Z}$$

(c) $X = \mathbb{RP}^2$



$$\Delta_0(X) = \langle v,w\rangle = \mathbb{Z}^2$$

$$\Delta_1(x) = \langle a,b,c\rangle = \mathbb{Z}^3$$

$$\Delta_2(X) = \langle U,L\rangle = \mathbb{Z}^2$$

$$\cdots \to 0 \longrightarrow \overset{\langle U,L\rangle}{\overset{\|}{\mathbb{Z}^2}} \xrightarrow{\partial_2} \overset{\langle a,b,c\rangle}{\overset{\|}{\mathbb{Z}^3}} \xrightarrow{\partial_1} \overset{\langle v,w\rangle}{\overset{\|}{\mathbb{Z}^2}} \xrightarrow{\partial_0} 0$$

$\partial_0 = 0$

$\partial_1(a) = \partial_1(b) = w-v$, $\quad \partial_1(c) = 0$

$\partial_2(U) = c+b-a$, $\quad \partial_2(V) = c+a-b$

$$H_0^{\Delta}(x) = \frac{\mathrm{Ker}(\partial_0)}{\mathrm{Im}(\partial_1)} = \frac{\langle v,w\rangle}{\langle w-v\rangle} = \frac{\langle v, v-w\rangle}{\langle w-v\rangle} \cong \mathbb{Z}$$

$$H_1^{\Delta}(x) = \frac{\mathrm{Ker}(\partial_1)}{\mathrm{Im}(\partial_2)}$$

$$\begin{aligned}
Ker(\partial_1) &= \{pa+qb+rc \mid \partial_1(pa+qb+rc)=0\} \\
&= \{pa+qb+rc \mid (p+q)(w-v)=0\} \\
&= \{pa+qb+rc \mid p=-q\} \\
&= \{pa-pb+rc\} = \langle a-b, c\rangle \\
&\cong \mathbb{Z}^2
\end{aligned}$$

$$\begin{aligned}
Im(\partial_2) &= \langle c+b-a, c+a-b\rangle \\
&= \langle c+a-b, 2c\rangle
\end{aligned}$$

$$\frac{Ker(\partial_1)}{Im(\partial_2)} = \frac{\langle a-b, c\rangle}{\langle c+a-b, 2c\rangle} = \frac{\langle c+a-b, c\rangle}{\langle c+a-b, 2c\rangle}$$

$$\cong \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2$$

$$H_2^{\Delta}(X) = \frac{Ker(\partial_2)}{Im(\partial_3)}$$

Now, $\partial_2(U) = c+b-a$

$\partial_2(L) = c+a-b$

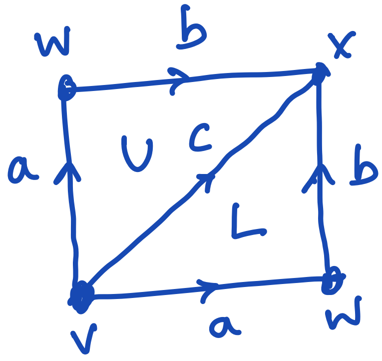
$\Rightarrow \partial_2$ is injective $\Rightarrow \mathrm{Ker}(\partial_2) = 0$

$$\Rightarrow \frac{\mathrm{Ker}(\partial_2)}{\mathrm{Im}(\partial_3)} = \frac{\mathbb{Z}^2}{\{0\}} \simeq \mathbb{Z}^2$$

(d) S^n – 2 copies of $\Delta^n(U,L)$ glued along the boundary by the identity.

$$H_n^{\Delta}(S^n) = \frac{\mathrm{Ker}(\partial n)}{\mathrm{Im}(\partial n{+}1)} = \frac{\langle U-L\rangle}{\{0\}} \simeq \mathbb{Z}$$

S^2



$$\Delta^0(S^2) = \langle v, w, x \rangle = \mathbb{Z}^3$$

$$\Delta^1(S^2) = \langle a, b, c \rangle = \mathbb{Z}^3$$

$$\Delta^2(S^2) = \langle U, L \rangle = \mathbb{Z}^2$$

$$\cdots \longrightarrow 0 \xrightarrow{\partial_3} \mathbb{Z}^2 \xrightarrow{\partial_2} \mathbb{Z}^3 \xrightarrow{\partial_1} \mathbb{Z}^3 \xrightarrow{\partial_0} 0$$

$\partial_0 = 0$

$\partial_1(a) = w - v$, $\partial_1(b) = x - w$, $\partial_1(c) = x - v$

$\partial_2(U) = \partial_2(L) = a + b - c$

$\partial_3 = 0$

$$H_0^{\Delta}(S^2) = \frac{\mathrm{Ker}(\partial_0)}{\mathrm{Im}(\partial_1)} = \frac{\langle u, v, x \rangle}{\langle u - v, v - x, x - u \rangle} \cong \frac{\mathbb{Z}^3}{\mathbb{Z}^3} \cong \{0\}$$

$$H_1^{\Delta}(S^2) = \frac{\mathrm{Ker}(\partial_1)}{\mathrm{Im}(\partial_2)} \cong \frac{\{0\}}{\mathrm{Im}(\partial_2)} \cong \{0\}$$

∂_1 is injective

$$H_2^{\Delta}(S^2) = \frac{\mathrm{Ker}(\partial_2)}{\mathrm{Im}(\partial_3)} = \frac{\mathbb{Z}}{\{0\}} \cong Z$$

$$\begin{aligned}
\mathrm{Ker}(\partial_2) &= \{pU+qL \mid \partial_2(pU+qL)=0\} \\
&= \{pU+qL \mid (p+q)(a+b-c)=0\} \\
&= \{pU+qL \mid p=-q\} \\
&= \langle U-L \rangle \cong \mathbb{Z}
\end{aligned}$$

Singular homology

Defn. A singular n-simplex is a continuous map $\sigma : \Delta^n \longrightarrow X$.

- We define $C_n(X)$ to be the free abelian group generated by the singular n-simplices in X.
- The elements of $C_n(x)$ are called singular n-chains, which are finite formal sums $\sum_i n_i \sigma_i$ for $n_i \in \mathbb{Z}$ and $\sigma_i : \Delta^n \longrightarrow X$.
- We define the boundary map $\partial_n : C_n(X) \longrightarrow C_{n-1}(X)$ by

$$\partial_n(\sigma) = \sum_i (-1)^i \sigma \Big| [v_0, \dots, \hat{v_i}, \dots v_n]$$

Here $\sigma|[v_0,\ldots,\hat{v_i},\ldots v_n]$ is regarded as a map $\Delta^{n-1} \longrightarrow X$ (i.e. a singular $(n-1)$-simplex).

Lemma: $\partial^2=0$ (i.e $\partial_n \circ \partial_{n+1} = 0$).

<u>Defn</u>. We define the <u>singular homology group</u> by

$$H_n(X) = \mathrm{Ker}(\partial_n) / \mathrm{Im}(\partial_{n+1})$$

<u>Remark</u>. (a) Its evident from the definition that homeomorphic spaces have isomorphic singular homology groups.

(b) $H_n(X)$ can also be viewed as a special case of $H_n^{\Delta}(X)$ in the following manner. Let $S(X)$ be the Δ-complex with one n-simplex Δ^n_σ for each singular simple $\sigma : \Delta^n \longrightarrow X$, with Δ^n_σ attached to the $(n-1)$-simplices (of $S(X)$) via the restrictions of σ to $\partial\Delta^n$.

Then $H_n^{\Delta}(S(X)) \cong H_n(X)$.

(c) The elements of $H_1(X)$ are represented by collections of oriented loops into X.

Prop. Corresponding to the decomposition of a space X into its path-components X_α, $\exists$ an isomorphism

$$H_n(X) \cong \bigoplus_\alpha H_n(X_\alpha).$$

Proof. A singular simplex has a path-connected image, so

$$C_n(X) = \bigoplus_\alpha C_n(X_\alpha).$$

Moreover, ∂_n preserves this decomposition.

Prop. If X nonempty and path-connected, then $H_0(X) \cong \mathbb{Z}$. Hence, for a any space X, $H_0(X)$ is a direct sum of $\mathbb{Z}$'s, one for each path component of X.

Proof. Since $\partial_0 = 0$, we have $H_0(X) = C_0(X)/\operatorname{Im}\partial_1$.

Define $\varepsilon : C_0(X) \longrightarrow \mathbb{Z}$ by $\varepsilon(\sum_i n_i \sigma_i) = \sum_i n_i$

Note that ε is a hom. which is surjective if $X \neq \emptyset$.

<u>Claim.</u> $\operatorname{Ker} \varepsilon = \operatorname{Im} \partial_1$

For a singular 1-simplex $\sigma : \Delta^1 \longrightarrow X$, we have

$$\varepsilon(\partial_1(\sigma)) = \varepsilon(\sigma|_{[v_1]} - \sigma|_{[v_0]})$$
$$= 1 - 1 = 0$$

$\Longrightarrow \operatorname{Im}(\partial_1) \subset \operatorname{Ker}(\varepsilon)$.

Now consider $\sigma = \sum_i n_i \sigma_i$ $\in \operatorname{Ker}(\varepsilon)$

Then $\varepsilon(\sigma) = \sum n_i = 0$

Note that σ_i's are essentially points of X.

Fix a basepoint $x_0 \in X$, and a path $\tau_i : I \longrightarrow X$ from x_0 to $\sigma_i(v_0)$.

Then viewing τ_i as a map $[v_0, v_1] \longrightarrow X$ (i.e. a singular 1-simplex), we have:

$$\partial \tau_i = \sigma_i - \sigma_0$$

Hence, $\partial(\sum n_i \tau_i) = \sum_i n_i \sigma_i - \sum_i n_i \sigma_0$

$$= \sum_i n_i \sigma_i$$

$$\Rightarrow \sigma = \sum_i n_i \sigma_i \in \mathrm{Im}(\partial)$$

Thus, we have $\mathrm{Ker}(\varepsilon) \subset \mathrm{Im}(\partial)$

$\blacksquare$

Prop: If X is a point, then $H_n(X) = 0$, for $n > 0$ and $H_0(X) \cong \mathbb{Z}$

Proof. In this case, $\exists!$ singular n-simplex σ_n for each n, and

$$C_n(X) = \langle \sigma_n \rangle \cong \mathbb{Z}$$

Moreover,

$$\partial_n(\sigma_n) = \sum_{i=0}^{n} (-1)^i \sigma_{n-1}$$

$$= \begin{cases} 0, & \text{if } n \text{ is odd} \\ \sigma_{n-1}, & \text{if } n \text{ is even} \end{cases}$$

Thus, we have the chain complex

$$\cdots \to \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} 0,$$

from which our assertion follows. ∎

Defn. Consider the augmented chain complex of a space $X \neq \emptyset$

$$\cdots \to C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

The homology associated with this complex are called reduced homology groups $\tilde{H}_n(X)$.

Lemma. For a space $X \neq \emptyset$, we have:

(a) $H_0(X) \cong \tilde{H}_0(X) \oplus \mathbb{Z}$

(b) $H_n(X) \cong \tilde{H}_n(X)$, for $n \geq 1$.

Proof (i) Since $\varepsilon \circ \partial_1 = 0$, we have $\mathrm{Im}(\partial_1) \subset \mathrm{Ker}\,\varepsilon$. So, ε induces a map $H_0(X) \xrightarrow{\bar{\varepsilon}} \mathbb{Z}$. Note that $\mathrm{Ker}(\bar{\varepsilon}) = \mathrm{Ker}(\varepsilon)/\mathrm{Im}(\partial_1) = \tilde{H}_0(X)$

$\Longrightarrow$ $H_0(X)/\tilde{H}_0(X) \cong \mathbb{Z}$, and the assertion in (i) follows.

(ii) This is apparent by definition.

Note. One can view the extra $\mathbb{Z}$ in the augmented chain complex as generated by the empty simplex $[\emptyset]$. Then ε becomes the usual boundary map as

$$\partial([v_0]) = [\hat{v}_0] = [\emptyset].$$

Homotopy Invariace

Proposition. A map $f: X \longrightarrow Y$ induces a homomorphism $f_*: H_n(X) \longrightarrow H_n(Y)$, for all n.

Proof. Consider the map $f_\#: C_n(X) \longrightarrow C_n(Y)$ defined by $\sigma \xmapsto{f_\#} f \circ \sigma$. for each singular n-simplex $\sigma: \Delta^n \longrightarrow X$, an then extended linearly to n-chains by

$$f_\#\left(\sum_\alpha n_\alpha \sigma_\alpha\right) = \sum_\alpha n_\alpha f_\#(\sigma_\alpha).$$

Thus, we obtain the following diagram

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1}(x) \arrow[r, "\partial"] \arrow[d, "f_\#"] & C_n(X) \arrow[r, "\partial"] \arrow[d, "f_\#"] & C_{n-1}(X) \arrow[r] \arrow[d, "f_\#"] & \cdots \\
\cdots \arrow[r] & C_{n+1}(Y) \arrow[r, "\partial"] & C_n(Y) \arrow[r, "\partial"] & C_{n-1}(Y) \arrow[r] & \cdots
\end{tikzcd}$$

<u>Claim</u>. Each square in this diagram commutes.

<u>Proof</u>. Note that

$$\begin{aligned}
(f_\# \circ \partial)(\sigma) &= f_\#\left(\sum_i (-1)^i \sigma|_{[v_0, \ldots \hat{v_i} \ldots v_n]}\right) \\
&= \sum_i (-1)^i (f \circ \sigma)|_{[v_0, \ldots \hat{v_i} \ldots v_n]} \\
&= (\partial \circ f_\#)(\sigma),
\end{aligned}$$

which establishes our claim.

Now suppose that $\partial\alpha = 0$ (i.e. α is a cycles). Then:

$$\partial(f_\# \alpha) = f_\#(\partial\alpha) = 0$$

$\Rightarrow$ $f_\#$ takes cycles to cycles ── ①

Moreover, since $f_\#(\partial\beta) = \partial(f_\#\beta)$, $f_\#$ takes boundaries to boundaries. ── ②

From ① & ②, it follows that $f_\#$ induces a homomorphism.

$$f_* : H_n(X) \longrightarrow H_n(Y)$$

∎

Defn. A $f_\# : C_n(X) \longrightarrow C_n(Y)$ as in the Proposition above is called a chain map.

Lemma

(i) For a composition of maps $X \xrightarrow{g} Y \xrightarrow{f} Z$, we have:

$$(f \circ g)_* = f_* \circ g_* .$$

(ii) For the identity map $id_X : X \longrightarrow X$, we have:

$$(id_X)_* = id_{H_n(X)} . \quad \square$$

Defn. A map $f : X \longrightarrow Y$ is said to be a *homotopy equivalence* if there exists a map $g : Y \longrightarrow X$ such that $(f \circ g) \simeq id_Y$ and $(g \circ f) \simeq id_X$.

Defn. Two spaces X & Y are *homotopically equivalent* $(X \simeq Y)$ if $\exists$ a homotopy equivalence $f : X \longrightarrow Y$. ,

Theorem. If two maps $f, g : X \longrightarrow Y$ are homotopic then $f_* = g_* : H_n(X) \longrightarrow H_n(Y)$.

Corollary (a) For a homotopy equivalence $f : X \longrightarrow Y$, we have $f_* : H_n(X) \longrightarrow H_n(Y)$ is an isomorphism.

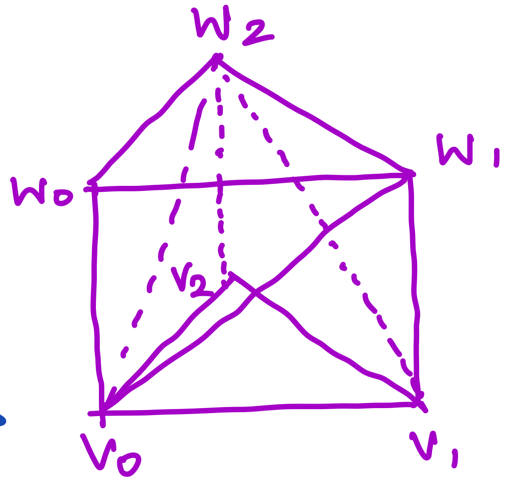
(b) If X is contractible (i.e. $X \simeq pt$), then

$$\tilde{H}_n(X) = \{0\}, \text{ for all } n.$$

Proof (of Theorem).

A vital ingredient in the proof is the subdivision of $\Delta^n \times I$ into $(n+1)$-simplices.

Let $\Delta^n \times \{0\} = [v_0, \ldots, v_n]$ and $\Delta^n \times \{1\} = [w_0, \ldots, w_n]$, where v_i and w_i have the same projection under $\Delta^n \times I \longrightarrow \Delta^n$.



Claim. $\Delta^n \times I$ is the union of the $(n+1)$- simplices $[v_0, \ldots, v_i, w_i, \ldots w_n]$

Proof (of claim). Note that the n-simplex $[v_0, \ldots, v_i, w_{i+1}, \ldots w_n]$ is the graph of the function $\varphi_i : \Delta^n \longrightarrow I$

defined by $\varphi_i(t_0,\dots t_n) = t_{i+1}+\dots+t_n$ in barycentric coordinates.

The simplex $[v_0,\dots v_i, w_i, \dots, w_n]$ projects homeomorphically to Δ^n under $\Delta^n \times I \longrightarrow \Delta^n$.

Since the graph of φ_i lies below graph of φ_{i-1} ($\because \varphi_i \leq \varphi_{i-1}$), the simplex $[v_0,\dots v_i, w_i,\dots w_n]$ bounded by these 2 graph is a true $(n+1)$-simplex.

From the inequalities,

$$0 = \varphi_n \leq \varphi_{n-1} \leq \dots \leq \varphi_{-1} \leq 1,$$

we see that $\Delta^n \times I$ is the union of the simplices $[v_0, \dots, v_i, w_i, \dots, w_n]$, which proves the claim.

Given a homotopy $F: X \times I \to Y$ from f to g, we define the prism operators

$$P : C_n(X) \longrightarrow C_{n+1}(Y)$$

by:

$$P(\sigma) = \sum_i (-1)^i F \circ (\sigma \times id) \big|_{[v_0, \dots v_i, w_i, \dots, w_n]}$$

where $\sigma : \Delta^n \longrightarrow X$ and $F \circ (\sigma \times id)$ is given by:

$$\Delta^n \times I \xrightarrow{\sigma \times id} X \times I \xrightarrow{F} Y$$

Then with a little bit of effort it can be verified that

$$\partial P = g_\# - f_\# - P\partial \quad (\text{Exercise})$$

Now, for a cycle $\alpha \in C_n(X)$, we have:

$$g_\#(\alpha) - f_\#(\alpha) = \partial P(\alpha),$$

since $\partial \alpha = 0$.

$\Rightarrow$ $g_\#(\alpha) - f_\#(\alpha)$ is a boundary, and hence

$g_*([\alpha]) = f_*([\alpha])$ in $H_n(Y)$.

$\blacksquare$

Relative homology groups

Defn. A sequence of homomorphisms

$$\cdots \to A_{n+1} \xrightarrow{\partial_{n+1}} A_n \xrightarrow{\partial_n} A_{n-1} \to \cdots \quad \leftarrow (*)$$

is said to be exact if $\mathrm{Ker}(\partial_n) = \mathrm{Im}(\partial_{n+1})$ $\forall n$.

Remark

(a) Note that $\partial_n \partial_{n+1} = 0$ ($\Leftrightarrow \mathrm{Im}(\partial_{n+1}) < \mathrm{Ker}(\partial_n)$) $\Rightarrow (*)$ is a chain complex.

(b) Since $\mathrm{Ker}(\partial_n) \subset \mathrm{Im}(\partial_{n+1})$, the associated homology groups are trivial.

Lemma.

(i) $0 \to A \xrightarrow{\alpha} B$ is exact iff $\ker(\alpha) = 0$ iff α is injective.

(ii) $A \xrightarrow{\alpha} B \to 0$ is exact iff $\operatorname{Im}(\alpha) = B$ iff α is surjective.

(iii) $0 \to A \xrightarrow{\alpha} B \to 0$ is exact iff α is an isom.

(iv) $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ is exact iff α is injective, β is surjective, and $\ker\beta = \operatorname{Im}\alpha$

$\Longleftrightarrow C \cong B/A$.

Defn. Given a space X and a subspace $A \subset X$, let

$$C_n(X,A) := C_n(X)/C_n(A)$$

Since $\partial : C_n(X) \longrightarrow C_{n-1}(X)$ and $\partial(C_n(A)) \subset C_{n-1}(A)$, $\exists$ a chain complex:

$$\cdots \longrightarrow C_{n+1}(X,A) \xrightarrow{\partial} C_n(X,A) \xrightarrow{\partial} C_{n-1}(X,A) \longrightarrow \cdots$$

called the chain complex of X relative A (or the pair (X,A)) whose associated homology groups are called relative homology groups $H_n(X,A)$.

Remark

(i) A class $[\alpha] \in H_n(X, A)$ is represented by a cycle $\alpha \in C_n(X)$ such that $\partial \alpha \in C_{n-1}(A)$.

(ii) A class $[\alpha] \in H_n(X, A)$ is trivial iff $\alpha = \partial \beta + \gamma$, for some $\beta \in C_{n+1}(X)$ and $\gamma \in C_n(A)$.

(iii) $H_n(X, A)$ (as we will show) measures the difference between the groups $H_n(X)$ and $H_n(A)$. Intuitively, it can be viewed as the homology of "X modulo A."

Theorem. For any pair (X, A), there exists a long exact sequence

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to H_{n-1}(X) \to \cdots,$$

where i_* is induced by the inclusion map $C_n(A) \longrightarrow C_n(X)$ and j_* is induced by the quotient map $j: C_n(X) \longrightarrow C_n(X)/C_n(A)$

Proof. We consider the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & & 0 \arrow[d] & \\
\cdots \arrow[r] & C_{n+1}(A) \arrow[r, "\partial_{n+1}"] \arrow[dd, "i"] & C_n(A) \arrow[rrr, "\partial_n"] \arrow[dd, "i"] & & & C_{n-1}(A) \arrow[r, "\partial_{n-1}"] \arrow[dd, "i"] & \cdots \\
 & & & & a \arrow[d, dotted, no head] & & \\
\cdots \arrow[r] & C_{n+1}(X) \arrow[r, "\partial_{n+1}"] \arrow[d, "j"] & C_n(X) \arrow[rrr, "\partial_n"] \arrow[d, "j"] & b \arrow[r, dashed] & \partial_n(b) & C_{n-1}(X) \arrow[r, "\partial_{n-1}"] \arrow[d, "j"] & \cdots \\
\cdots \arrow[r] & C_{n+1}(X,A) \arrow[r, "\partial_{n+1}"] \arrow[d] & C_n(X,A) \arrow[rrr, "\partial_n"] \arrow[d] & c \arrow[u, dotted] & & C_{n-1}(X,A) \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & & & 0 &
\end{tikzcd}$$

The commutativity of this diagram ensures the i_* and j_* are induced on homology.

Now, we define a boundary map $\bar{\partial} : H_n(X,A) \longrightarrow H_{n-1}(A)$.

Consider a class $[c] \in H_n(X,A)$

Since $c \in C_n(X)/C_n(A)$ and $\mathring{j}$ is surjective, $\exists$ a $b \in C_n(X)$ such that $\mathring{j}(b) = c$.

Then $\partial_n(b) \in C_{n-1}(X)$. Since $\mathring{j} \circ \partial_n(b) = \partial_n(\mathring{j}(b)) = \partial_n(c) = 0$, we have $\partial_n(b) \in \operatorname{Ker}(\mathring{j})$

As $\operatorname{Ker}(\mathring{j}) = \operatorname{Im}(i)$, $\exists$ $a \in C_{n-1}(A)$ such that $\mathring{i}(a) = \partial_n(b)$.

Moreover, we have:

$$\mathring{i}(\partial_{n-1}(a)) = \partial_{n-1}(i(a)) = \partial_{n-1}(\partial_n(b)) = 0$$

$\Rightarrow \partial_{n-1}(a) = 0$ ($\because$ $\mathring{i}$ is injective).

Thus, we define a map

$$\bar{\partial} : H_n(X,A) \longrightarrow H_{n-1}(A)$$

by $\bar{\partial}([c]) = [a]$.

Claim. $\bar{\partial}$ is a well-defined homomorphism.

Proof (of claim).

Well-definedness:

First, we note that a is uniquely determined by ∂b since i is injective.

Suppose we had chosen a different b' such that $j(b') = c$.

Then $j(b) = j(b') = c \implies$

$j(b'-b) = 0 \Rightarrow b'-b' \in \operatorname{Ker} j = \operatorname{Im} i$

$\Rightarrow b'-b = i(a) \Rightarrow b' = b + i(a)$

Changing b with $b+i(a)$ simply replaces a to a homologous element $a + \partial a'$

$$\left[i(a + \partial_{n-1}(a')) = i(a) + i(\partial_{n-1}(a')) = \partial_n(b) + \partial_{n-1}(i(a')) \right]$$

Similarly a different choice for c within its homology class leaves ∂b and a unchanged (Check!).

Thus $\bar{\partial}$ is well-defined.

Finally, the fact that i, j and ∂_n are homomorphisms would imply that $\bar{\partial}$ is a homomorphisms.

Exactness of the LES

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\bar{\partial}} H_{n-1}(A) \to \cdots$$

Im$(i_*) \subset$ Ker(j_*). This follows from the fact that $(j \circ i) = 0$ ($\Rightarrow j_* \circ i_* = 0$)

Im$(j_*) \subset$ Ker$(\bar{\partial})$. By definition, we have $\bar{\partial} = i^{-1} \circ \partial \circ j^{-1} \Longrightarrow \bar{\partial} \circ j = i^{-1} \circ \partial = 0$ on cycles rep $H_n(X,A)$.

$\mathrm{Im}(\bar{\partial}) \subset \mathrm{Ker}(i_*)$ By definition, we have $i \circ \bar{\partial} = \partial \circ j^{-1} = 0$ (on cycles rep $H_n(X,A)$.

$\mathrm{Ker}(j_*) \subset \mathrm{Im}(i_*)$

Let $[b] \in \mathrm{Ker}(j_*)$. Then b is a cycle in $C_n(X)$ such that

$$j(b) \in \partial_{n+1}\left(\frac{C_{n+1}(X)}{C_{n+1}(A)}\right) \Longrightarrow$$

$$\exists\, c' \in \frac{C_{n+1}(X)}{C_{n+1}(A)} \text{ s.t } \partial_{n+1}(c') = j(b)$$

Moreover, as j is surjective, $\exists$ $b' \in C_{n+1}(X)$ such that $j(b') = c'$.

Now,
$$\begin{aligned} j(b - \partial_{n+1}(b')) &= j(b) - j \circ \partial_{n+1}(b') \\ &= j(b) - \partial_{n+1}(j(b')) \\ &= j(b) - j(b) = 0 \end{aligned}$$

$\Rightarrow b - \partial_{n+1}(b') = i(a)$, for some $a \in C_n(A)$.

Now $i(\partial_n(a)) = \partial_n(i(a))$

$$= \partial_n(b - \partial_{n+1}(b'))$$

$$= \partial_n(b) = 0$$

$\Rightarrow \partial_n(a) = 0$ ($\because$ i is injective)

Finally, $i_*([a]) = [b - \partial_{n+1}(b')]$

$$= [b]$$

$$\Rightarrow \mathrm{Ker}(j_*) \subset \mathrm{Im}(i_*)$$

$\underline{\mathrm{Ker}(\bar{\partial}) \subset \mathrm{Im}(j_*)}$

Let $[c] \in \mathrm{Ker}(\bar{\partial})$. Then as seen earlier, $\bar{\partial}([c]) = [a] = [0]$ $\Rightarrow a \in \mathrm{Im}(\partial_n) \Rightarrow a = \partial_n(a')$ for some $a' \in C_n(A)$.

Now,

$$\partial_n(b - i(a')) = \partial_n(b) - \partial_n(i(a'))$$
$$= \partial_n(b) - i(\partial_n(a'))$$
$$= \partial_n(b) - i(a)$$
$$= 0 \text{ (by defn)}$$

$\Rightarrow$ $b - i(a')$ is a cycle.

Moreover, $j(b - i(a'))$

$$= j(b) - j(i(a'))$$
$$= j(b) = c$$

$\Rightarrow$ $j_*([b - i(a')]) = [c]$

$\Rightarrow$ $\mathrm{Ker}(\bar{\partial}) \subset \mathrm{Im}(j_*)$

$\mathrm{Ker}(i_*) \subset \mathrm{Im}(\bar\partial)$

$$\cdots \to H_n(X,A) \xrightarrow{\bar\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(X) \to \cdots$$

Let $[a] \in \mathrm{Ker}(i_*)$. Then
$i(a) \in \partial_n(C_n(X)) \Longrightarrow$
$i(a) = \partial_n(b)$, for some
$b \in C_n(X)$

Then
$$\partial_n(j(b)) = j(\partial_n(b))$$
$$= j(i(a)) = 0$$

$\Longrightarrow j(b)$ is a cycle.

Thus, $\bar\partial(j(b)) = [a]$. ■

Remark. $H_n(X,A)$ measures the difference between the groups $H_n(X)$ and $H_n(A)$.

In particular, if $H_n(X,A)$ for all n, then $H_n(A) \underset{\cong}{\xrightarrow{i_*}} H_n(X)$

Defn. A space X and a closed subspace $A \subset X$ are said to form a good pair (X,A) if A has a nbhd in X that deformation retracts onto A.

Theorem. If (X,A) form a good pair of spaces, then $\exists$ a LES:

$$\to \tilde{H}_n(A) \xrightarrow{i_*} \tilde{H}_n(X) \xrightarrow{j_*} \tilde{H}_n(X/A) \xrightarrow{\partial} \tilde{H}_{n-1}(X) \to \cdots,$$

where i is the inclusion and j is induced the quotient $X \longrightarrow X/A$.

Remark. A cell-complex X and a subcomplex $A \subset X$ always form a good pair.

Corollary. $\tilde{H}_n(S^n) \cong \mathbb{Z}$ and $\tilde{H}_i(S^n) = 0$, for $i \neq n$.

Proof Consider the CW-pair (D^n, S^{n-1}). From the LES of reduced homology groups, we have :

$$\longrightarrow \overset{0}{\overset{\shortparallel}{\tilde{H}_i(D^n)}} \xrightarrow{j_*} \tilde{H}_i(D^n/S^{n-1}) \xrightarrow{\bar{\partial}} \tilde{H}_i(S^{n-1})$$

$$\xrightarrow{i_*} \underset{\underset{0}{\shortparallel}}{\tilde{H}_{i-1}(D^n)} \longrightarrow$$

$$\Rightarrow \tilde{H}_i(S^n) \xrightarrow[\cong]{\bar{\partial}} \tilde{H}_{i-1}(S^{n-1}),$$ for all $i>0$

If $i=n$, then

$$\tilde{H}_i(S^n) \cong \tilde{H}_0(S^0) \cong \mathbb{Z}$$

$$(\because H_0(S^0) \cong \mathbb{Z} \oplus \mathbb{Z})$$

If $n>i$, then:

$$\tilde{H}_i(S^n) \cong \tilde{H}_0(S^{n-i}) \cong \{0\}$$

If $n<i$, then:

$$\tilde{H}_i(S^n) \cong \tilde{H}_{i-n}(S^0) \cong \{0\}$$

$\blacksquare$

Corollary (No retraction theorem). There exists no retraction

$$D^n \longrightarrow \partial D^n.$$

Proof. Suppose $\exists$ a retraction $r: D^n \longrightarrow \underset{\substack{\parallel \\ S^{n-1}}}{\partial D^n}$. Then $r \circ i = id_{S^{n-1}}$,

and so the composition

$$\tilde{H}_{n-1}(S^{n-1}) \xrightarrow{i_*} \tilde{H}_{n-1}(D^n) \xrightarrow{r_*} \tilde{H}_{n-1}(S^{n-1})$$

equals $id_{\tilde{H}_{n-1}(S^{n-1})} = id_{\mathbb{Z}}$

This is a contradiction as

$\tilde{H}_{n-1}(D^n) = 0$. ∎

Corollary (Brouwer's Fixed Point Theorem). Every (continuous) map $f : D^n \longrightarrow D^n$ has a fixed point.

Proof.

Suppose that $f : D^n \longrightarrow D^n$ has no fixed. Then the map $r : D^n \longrightarrow S^n : v \longmapsto \frac{v - f(v)}{\|v - f(v)\|}$ defines a retraction $\#$ ∎

Remark. There exists a LES of reduced homology analogous to the LES of homology group. This is obtained by taking the

SES $0 \to C_n(A) \to C_n(X) \to C_n(X,A) \to 0$ in non-negative dimensions and the SES $0 \to \mathbb{Z} \xrightarrow{id} \mathbb{Z} \to 0 \to 0$ in dimension -1. In particular, this would imply that $H_n(X,A) = \tilde{H}_n(X,A)$ for all n, when $A \neq \emptyset$.

__Example__. Consider the LES of reduced homology groups of the pair (X, x_0), where $x_0 \in X$. We have

$$\to \tilde{H}_n(x_0) \to \tilde{H}_n(X) \to \tilde{H}_n(X, x_0) \to \tilde{H}_{n-1}(x_0)$$

Since $\tilde{H}_n(x_0) = 0 \; \forall n$, we have

$$\tilde{H}_n(X) \cong \tilde{H}_n(X, x_0) = H_n(X, x_0)$$

Example By considering the LES of the pair $(D^n, \partial D^n)$, we have $H_i(D^n, \partial D^n) \xrightarrow{\partial} \tilde{H}_{i-1}(S^{n-1})$ are isomorphisms for all $i > 0$.

Consequently, we have:

$$H_i(D^n, \partial D^n) \cong \begin{cases} \mathbb{Z}, & \text{if } i = n \\ 0, & \text{otherwise} \end{cases}$$

Remark

A map $f: X \longrightarrow Y$ with $f(A) \subset B$ (i.e. $f: (X,A) \longrightarrow (Y,B)$) induces homs $f_\#: C_n(X,A) \longrightarrow C_n(Y,B)$ such that:

(a) $f_\# \partial = \partial f_\#$ for relative chains

(b) For $g \simeq f$ (via maps of pairs $(X,A) \longrightarrow (Y,B)$), we have:

$$\partial P + P \partial = g_\# - f_\# ,$$

where $P: C_n(X,A) \longrightarrow C_{n+1}(Y,B)$ is the induced prism operator.

Proposition. If two maps $f, g: (X,A) \longrightarrow (Y,B)$ are homotopic through maps of pairs $(X,A) \to (Y,B)$, then $f_* = g_*: H_n(X,A) \longrightarrow H_n(Y,B)$.

Proposition. For a triple (X,A,B) of spaces with $B \subset A \subset X$, $\exists$ a LES

$$\cdots \to H_n(A,B) \to H_n(X,B) \longrightarrow H_n(X,A) \longrightarrow H_{n-1}(A,B) \to \cdots$$

associated with SES

$$0 \to C_n(A,B) \to C_n(X,B) \longrightarrow C_n(X,A) \longrightarrow 0.$$

Excision

Theorem (Excision). Given subspaces $Z \subset A \subset X$ such that $\bar{Z} \subset A^\circ$, then the inclusion $(X-Z, A-Z) \hookrightarrow (X, A)$ induces isomorphisms $H_n(X-Z, A-Z) \longrightarrow H_n(X,A)$ for all n. Equivalently, for subspaces $A, B \subset X$ such that $X = A^\circ \cup B^\circ$, the inclusion $(B, A\cap B) \hookrightarrow (X, A)$ induces isomorphisms:

$$H_n(B, A\cap B) \longrightarrow H_n(X,A), \text{ for all } n.$$

Proof: Exercise (May be covered later)

Proposition. For good pairs (X,A) the quotient map $q:(X,A)\to(X/A, A/A)$ induces isomorphisms

$$q_*: H_n(X,A)\to H_n(X/A, A/A)\cong\tilde{H}_n(X/A)$$

for all n.

Proof. Let V be a nbhd of A that deformation retracts onto X. We have the following commutative diagram.

$$\begin{tikzcd}
H_n(X,A) \arrow[r, "\cong_1"] \arrow[d, "q_*"] & H_n(X,V) \arrow[d, "q_*"] & H_n(X-A, V-A) \arrow[l, "\cong_3"'] \arrow[d, "q_*", "\cong_5"'] \\
H_n(X/A, A/A) \arrow[r, "\cong^2"] & H_n(X/A, V/A) & H_n\left(\frac{X}{A}-\frac{A}{A}, \frac{V}{A}-\frac{A}{A}\right) \arrow[l, "\cong_4"']
\end{tikzcd}$$

$\cong_1$. Follows from LES of the triple (X,V,A).

$$\cdots \to \overset{\overset{0}{\parallel}}{H_n(V,A)} \to H_n(X,V) \xrightarrow{\cong} H_n(X,A) \to \underset{}{\overset{\overset{0}{\parallel}}{H_{n-1}(V,A)}} \to \cdots$$

$H_n(V,A) = 0$ $\forall n$ as V def retracts onto A.

$\cong_2$ Follows from an analogous the argument by considering the triple $(X/A, V/A, A/A)$ and the fact that V/A def. ret. onto A/A.

$\cong_3$ & $\cong_4$: Follow from Excision Theorem.

$\cong_5$: Since $q|_{X-A}$ is a homeo, the isomorphism follows.

The assertion now follows from

the commutativity of the diagram

Examples

(a) Let (X, A) be a good pair, at let the cone CA of A be defined by $CA = A \times I / A \times \{0\}$.

Then:

$$\tilde{H}_n(X \cup CA) \cong H_n(X \cup CA, CA)$$ [LES of pair $(X \cup CA, CA)$]

$$\cong H_n(X \cup CA - \{p\}, X \cup CA - \{p\})$$ [Excision]

$$\cong H_n(X/A)$$ [$CA - \{p\}$ deformation retracts onto A]

Example (b) We wish to find the explicit cycles representing $H_n(D^n, \partial D^n)$. We may view $(D^n, \partial D^n)$ as the pair $(\Delta^n, \partial\Delta^n)$.

Claim. The identity $i_n : \Delta^n \longrightarrow \Delta^n$ (viewed as a singular n-simplex) is a cycle generating $H_n(\Delta^n, \partial\Delta^n)$.

Proof. i_n is clearly a cycle as we are considering $H_n(\Delta^n, \partial\Delta^n)$.

Our assertion holds trivially for $n=0$.

Assume the result holds for $n-1$.

For the inductive step, let $\Lambda \subset \Delta^n$ be the union of all but one of the $(n-1)$-dimensional faces of Δ^n. Note that:

(i) Δ^n deformation retracts onto $\Lambda \Rightarrow (\Delta^n, \Lambda) \simeq (\Lambda, \Lambda)$.

(ii) The inclusion $\Delta^{n-1} \hookrightarrow \partial\Delta^n$ as the face not contained in Λ induces homeomorphisms $\Delta^{n-1}/\partial\Delta^{n-1} \approx \partial\Delta^n/\Lambda$.

Now consider the following isomorphisms:

$$H_n(\Delta^n, \partial\Delta^n) \xrightarrow[\partial]{\cong_1} H_{n-1}(\partial\Delta^n, \Lambda) \xleftarrow{\cong_2} H_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1})$$

$\cong_1$: Follows from the LES of the triple $(\Delta^n, \partial\Delta^n, \Lambda)$ an (i) above, as $H_i(\Delta^n, \Lambda) = 0$, for all i.

$\cong_2$ Follows from the preceding Proposition and (ii).

By our induction hypothesis $H_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1}) = \langle i_{n-1} \rangle$. The assertion now follows from the fact that $\bar{\partial}(i_n) = \pm i_{n-1}$.
Regarding $S^n = \Delta_1^n \cup \Delta_2^n$ and applying a similar reasoning $H_n(S^n) = \langle \Delta_1^n - \Delta_2^n \rangle$

Corollary. If a CW-complex X is a union of subcomplexes A and B, then the inclusion $(B, A \cap B) \hookrightarrow (X, A)$ induces isomorphisms $H_n(B, A \cap B) \to H_n(X, A)$, for all n.

Proof. It follows directly from

the Proposition and the fact that $B/A \cap B \approx X/A$. ■

Corollary. For a wedge sum $\bigvee_{\alpha \in J} X_\alpha = \bigsqcup_{\alpha \in J} X_\alpha / \{x_\alpha : \alpha \in J\}$ with each pair (X_α, x_α) forming a good pair, the inclusions $i_\alpha : X_\alpha \hookrightarrow \bigvee_{\alpha \in J} X_\alpha$ induce an isomorphism

$$\bigoplus_{\alpha \in J} (i_\alpha)_* : \bigoplus_{\alpha \in J} \tilde{H}_n(X_\alpha) \to \tilde{H}_n\left(\bigvee_{\alpha \in J} X_\alpha\right)$$

Proof. Follows immeadietly by considering the pair $\left(\bigsqcup_{\alpha \in J} X_\alpha, \{x_\alpha : \alpha \in J\}\right)$ in the Proposition and the fact that $\tilde{H}_n(X_\alpha) \cong H_n(X_\alpha, x_\alpha)$. ■

Theorem (Brouwer, 1910). If nonempty open set $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ are homeomorphic, then $m = n$.

Proof. For $x \in U$, we have:

$$H_k(U, U - \{x\}) \cong H_k(\mathbb{R}^m, \mathbb{R}^m - \{x\})$$
(by excision)

$$\cong \tilde{H}_{k-1}(\mathbb{R}^m - \{x\})$$
(LES of pair $(\mathbb{R}^m, \mathbb{R}^m - \{x\})$)

$$\cong \tilde{H}_{k-1}(S^{m-1})$$
$(\mathbb{R}^m - \{x\} \simeq S^{m-1})$

$$\cong \begin{cases} \mathbb{Z}, & \text{if } k = m \\ 0, & \text{otherwise.} \end{cases}$$

Now, suppose that $f : U \longrightarrow V$ is a homeomorphism. Then f induces an isomorphism $H_k(U, U-\{x\}) \xrightarrow{f_*} H_k(V, V-\{h(x)\})$ for all k. Hence, it follows that $m=n$.

<u>Remark</u>. Given a map $f:(X,A)\to(Y,B)$, there exists a commutative diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(A) \arrow[r, "i_*"] \arrow[d, "f_*"] & H_n(X) \arrow[r, "j_*"] \arrow[d, "f_*"] & H_n(X,A) \arrow[r, "\partial"] \arrow[d, "f_*"] & H_{n-1}(A) \arrow[r] \arrow[d, "f_*"] & \cdots \\
\cdots \arrow[r] & H_n(B) \arrow[r, "i_*"] & H_n(X) \arrow[r, "j_*"] & H_n(X,B) \arrow[r, "\partial"] & H_{n-1}(B) \arrow[r] & \cdots
\end{tikzcd}$$

This property is called naturality, and it follows from the commutativity of the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & C_n(A) \arrow[r, "i"] \arrow[d, "f_\#"'] & C_n(X) \arrow[r, "j"] \arrow[d, "f_\#"] & C_n(X,A) \arrow[r] \arrow[d, "f_\#"] & 0 \\
0 \arrow[r] & C_n(B) \arrow[r, "i"] & C_n(Y) \arrow[r, "j"] & C_n(Y,B) \arrow[r] & 0
\end{tikzcd}$$

(which is obvious) and fact that

$f_\# \partial = \partial f_\#$.

In a similar manner, there also exists a commutative diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & \tilde{H}_n(A) \arrow[r, "i_*"] \arrow[d, "f_*"] & \tilde{H}_n(X) \arrow[r, "q_*"] \arrow[d, "f_*"] & \tilde{H}_n(X/A) \arrow[r, "\partial"] \arrow[d, "f_*"] & \tilde{H}_{n-1}(A) \arrow[r] \arrow[d, "f_*"] & \cdots \\
\cdots \arrow[r] & \tilde{H}_n(B) \arrow[r, "i_*"] & \tilde{H}_n(X) \arrow[r, "q_*"] & \tilde{H}_n(X/B) \arrow[r, "\partial"] & \tilde{H}_{n-1}(B) \arrow[r] & \cdots
\end{tikzcd}$$

Equivalence of simplicial and singular homology

Let X be a Δ-complex and A a subcomplex. Then $H_n^\Delta(X,A)$ is defined by considering the relative chain group $\Delta_n(X,A) = \Delta_n(X)/\Delta_n(A)$

There is a canonical homomorphism

$$H_n^\Delta(X,A) \longrightarrow H_n(X,A)$$

induced by natural chain map $\Delta_n(X,A) \to C_n(X,A)$ sending each n-simplex Δ^n_α characteristic map $\sigma^n_\alpha : \Delta^n_\alpha \longrightarrow X$ defined by the composition:

$$\Delta^n_\alpha \hookrightarrow X^{n-1} \sqcup_\alpha \Delta^n_\alpha \longrightarrow X^n \hookrightarrow X$$

(σ_α is in essence the composition of the attachin map with the quotient map)

Theorem. Let (X,A) be a Δ-complex pair. Then the homomomorphism $H_n^{\Delta}(X,A) \longrightarrow H_n(X,A)$ is an isomorphism for each n.

Proof. We first consider the case X is finite-dimensional and $A = \emptyset$. We have the following commutative diagram:

$$\begin{tikzcd}
H_{n+1}^{\Delta}(X^K, X^{K-1}) \arrow[r] \arrow[d, "\alpha"] & H_n^{\Delta}(X^{K-1}) \arrow[r] \arrow[d, "\beta"] & H_n^{\Delta}(X^K) \arrow[r] \arrow[d, "\gamma"] & H_n^{\Delta}(X^K, X^{K-1}) \arrow[r] \arrow[d, "\delta"] & H_{n-1}^{\Delta}(X^{K-1}) \arrow[d, "\varepsilon"] \\
H_{n+1}(X^K, X^{K-1}) \arrow[r] & H_n(X^{K-1}) \arrow[r] & H_n(X^K) \arrow[r] & H_n(X^K, X^{K-1}) \arrow[r] & H_{n-1}(X^{K-1})
\end{tikzcd}$$

Now, we make the following observations:

(a) $$\Delta_n(X^K, X^{K-1}) = \begin{cases} 0, & \text{if } n \neq K \\ \langle\{K\text{-simplices}\}\rangle, & \text{if } n = K \end{cases}$$

Thus, $H_n^{\Delta}(X^K, X^{K-1})$ has the same description.

(b) The characteristic map $\sigma_K^{\alpha} : \Delta_\alpha^K \to X$ induce:

$$\Phi : \coprod_\alpha (\Delta_\alpha^K, \partial\Delta_\alpha^K) \longrightarrow (X^K, X^{K-1})$$

The Φ induces a homeomorphism of quotient spaces:

$$\bigsqcup_\alpha \Delta^k_\alpha / \bigsqcup_\alpha \partial\Delta^k_\alpha \approx X^k / X^{k-1},$$

and hence an isomorphism of homology groups. Consequently, from the fact that $H_k(\Delta^k, \partial\Delta^k) = \langle i_{\Delta^k} \rangle$, we have:

$$H_n(X^k, X^{k-1}) = \begin{cases} 0, & \text{if } n \neq k \\ \left\langle \begin{matrix}\text{Relative cycles} \\ \text{given by char} \\ \text{map } \sigma^n_\alpha\end{matrix} \right\rangle, & \text{if } n = k \end{cases}$$

From (a) and (b), we have α and δ are isomorphisms. Moreover, by induction β and ε are also isomorphism.

Finally, we appeal to the following basic algebraic lemma:

The Five-Lemma. In a commutative diagram of abelian groups

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "\alpha"] & B \arrow[r, "j"] \arrow[d, "\beta"] & C \arrow[r, "k"] \arrow[d, "\gamma"] & D \arrow[r, "l"] \arrow[d, "\delta"] & E \arrow[d, "\varepsilon"] \\
A' \arrow[r, "i'"] & B' \arrow[r, "j'"] & C' \arrow[r, "k'"] & D' \arrow[r, "l'"] & E'
\end{tikzcd}$$

if the two rows are exact and α, β, δ, and ε are isomorphisms, then γ is an isomorphism.

Thus, from the five-lemma, it follows that γ is an isomorphism.

For the infinite dimensional case, we first make the following claim.

Claim. A compact set in X can meet only finitely many open simplices of X.

Proof(of claim). Suppose we assume that a compact set C intersected infinitely many open simplices $\Delta_i^{k_i}$

It would then contain an infinite sequence $\{x_i\}$ each lying in a different open simplex.

Then consider the sets $U_i = X - \bigcup_{j \neq i} \{x_i\}$

Note that each $(\sigma_i^{k_i})^{-1}(U_i)$ is open.

Thus, $\{U_i\}$ forms an open cover for C with no finite subcover $\#$

We use this claim to show that $H_n^\Delta(X) \longrightarrow H_n(X)$ is an isomorphism.

We only show the argument for

surjectivity as the injectivity follows along similar lines.

Consider a class $[z] \in H_n(X)$ represented by a singular n-cycle z. As z is a linear combination of finitely many singular simplices each with compact image. Thus z meets only finitely many open simplices in X and hence $z \in X^k$ for some k. Now the surjectivity of the map follows from the fact that $H_n^{\Delta}(X^k) \longrightarrow H_n(X^k)$ is an isomorphism.

For the case when $A \neq \emptyset$, we consider the following commutative diagram:

$$\begin{tikzcd}
H^{\Delta}_n(A) \arrow[r] \arrow[d, "\alpha'"] & H^{\Delta}_n(X) \arrow[r] \arrow[d, "\beta'"] & H^{\Delta}_n(X,A) \arrow[r] \arrow[d, "\gamma'"] & H^{\Delta}_{n-1}(A) \arrow[r] \arrow[d, "\delta'"] & H^{\Delta}_{n-1}(X) \arrow[d, "\varepsilon'"] \\
H_n(A) \arrow[r] & H_n(X) \arrow[r] & H_n(X,A) \arrow[r] & H_{n-1}(A) \arrow[r] & H_{n-1}(X)
\end{tikzcd}$$

Now α', β', δ', and ε' are isomorphisms from the case $A = \emptyset$. Therefore, γ' is an isomorphism from the five-lemma ∎

Defn. The number of $\mathbb{Z}$ summands in $H_n(X)$ is called the nth Betti number and the orders of its finite cyclic summands are called torsion coefficients.

Applications of homology

Defn. For a map $f: S^n \longrightarrow S^n$, the induced homomorphism $f_*: \tilde{H}_n(S^n) \longrightarrow \tilde{H}_n(S^n)$ is an isomorphism $\mathbb{Z} \to \mathbb{Z}$. Hence, $\exists\, d \in \mathbb{Z}$ such that $f_*(\alpha) = d\alpha$, for each $\alpha \in \tilde{H}_n(S^n)$. This integer d is called degree of f, denoted by $\deg(f)$.

Proposition (Properties of deg)

(a) $\deg(id) = 1$
(b) If f is surjective, then $\deg(f) = 0$.
(c) If $f \simeq g$, then $\deg(f) = \deg(g)$.
(d) $\deg(f \circ g) = \deg(f)\deg(g)$

(e) If f is a reflection of S^n fixing the points in a subsphere S^{n-1}, then $\deg(f) = -1$.

(f) The antipodal map $a: S^n \longrightarrow S^n$ has degree $(-1)^{n+1}$.

(g) If $f: S^n \longrightarrow S^n$ has no fixed points, then $\deg(f) = (-1)^{n+1}$.

Proof

(a) This is because $(id_{S^n})_*$

$= id_{\tilde{H}_n(S^n)}$.

(b) Suppose that $\exists\ x_0 \in S^n \setminus f(S^n)$, the f is the composition:

$$S^n \xrightarrow{f} S^n - \{x_0\} \overset{i}{\hookrightarrow} S^n$$

The assertion now follows

from the fact that $H_n(S^n - \{0\}) = 0$

(c) We know that if $f \simeq g$, then $f_* = g_*$. Therefore,

$$\deg(f) = \deg(g).$$

(Note that the converse of this statement is due to Hopf. 1925).

(d) This follows from the fact that $(f \circ g)_* = f_* \circ g_*$.

(e) We have seen that S has a Δ-complex structure with 2 n-simplices Δ_1^n, Δ_2^n attached along $\partial\Delta^n_i$, and that

$$H_n(S^n) = \langle \Delta_1^n - \Delta_2^n \rangle.$$

A reflection such as f would swap Δ_1^n with Δ_2^n, and so we have that $\Delta_1^n - \Delta_2^n \xmapsto{f_*} \Delta_2^n - \Delta_1^n$.

Thus, $\deg(f) = -1$.

(f) This is a direct consequence of the fact that a is a composition of $(n+1)$ reflections.

(g) If $f : S^n \to S^n$ has no fixed point, then the map

$$H : S^n \times I \to S^n : (x,t) \overset{H}{\longmapsto} \frac{(1-t)f(x) - tx}{\|(1-t)f(x) - tx\|}$$

defines a homotopy from f to a.

Thus, $\deg(f) = (-1)^{n+1}$.

Theorem S^n has continuous nonvanishing tangent vector field iff n is odd.

Proof. Let $v : S^n \longrightarrow \mathbb{R}^n$ be a nonvanishing tangent vector field. Since $v(x) \neq 0$ $(\forall x)$ we may normalize v by replacing v by $\frac{v}{\|v\|}$. Then, the map $F : S^n \times [0, \pi] \longrightarrow S^n$ defined by

$$F(x,t) = (\cos t)v(x) + (\sin t)v(x)$$

is a homotopy from id_{S^n} to a. Hence, we have that $(-1)^{n+1} = 1$, or n is odd.

Conversely, if n is odd, then $v(x_1, \dots, x_{2K-1}, x_{2K}) = (-x_2, x_1, \dots -x_{2K}, x_{2K-1})$ is a non-vanishing tangent vector field on S^n $\left(\because \begin{array}{l} \|v(x)\| = 1 \text{ and} \\ \langle x, v(x) \rangle = 0 \end{array} \right)$.

Proposition. $\mathbb{Z}_2$ is the only nontrivial group that can act freely on S^n if n is even.

Proof. An *action of a group* G on a space is defined to be a homomorphism $G \xrightarrow{\varphi} \text{Homeo}(X)$. Such an action is said to be *free* if $\varphi(g)$ has no fixed points for each $g \in G$.

Now the map $\deg : \mathrm{Homeo}(X) \to \{\pm 1\}$ induces a map $d : G \to \overset{\mathbb{Z}_2}{\overset{\shortparallel}{\{\pm 1\}}}$, where $d = \deg \circ \varphi$. Clearly, d is a homomorphism $\mathrm{Ker}(\varphi) = \{1\}$, if n is even. Thus, $G \subset \mathbb{Z}_2$ ■

Remark. Let $f : S^n \to S^n$ have the property that for some point $y \in S^n$, $f^{-1}(y) = \{x_1, x_2, \dots, x_m\}$. Let $U_i \ni x_i$ be a nbhd such that $f(U_i) \subset V$, where V is a nbhd of y.

Then we have the following commutative diagram.

$$\begin{tikzcd}
 & H_n(U_i, U_i - \{x_i\}) \arrow[r, "f_*"] \arrow[dl, "\cong_1"'] \arrow[d, "k_i"] & H_n(V, V - y) \arrow[d, "\cong_2"] \\
H_n(S^n, S^n - x_i) & H_n(S^n, S^n - f^{-1}(y)) \arrow[l, "p_i"'] \arrow[r, "f_*"] & H_n(S^n, S^n - y) \\
 & H_n(S^n) \arrow[ul, "\cong_3"] \arrow[u, "j"'] \arrow[r, "f_*"] & H_n(S^n) \arrow[u, "\cong_4"']
\end{tikzcd}$$

Here k_i, p_i are induced by inclusions. $\cong_1$ and $\cong_2$ follow from Excision, while $\cong_3$ and $\cong_4$ follow from the exact sequence of pairs. Thus, the f_* (on top) becomes an isomorphism since

$$H_n(U_i, U_i - x_i) \cong H_n(V, V - y) \cong H_n(S^n) \cong \mathbb{Z}$$

In particular, f_* is multiplication by an integer called the <u>local degree of f at x_i</u>. (denoted by $\deg(f|x_i)$).

Proposition $\deg(f) = \Sigma_i \deg(f|x_i)$.

Proof By Excision, it follows that $H_n(S^n, S^n - f^{-1}(y)) \cong \bigoplus_{i=1}^{m} H_n(U_i, U_i - x_i)$

$$\cong \bigoplus_{i=1}^{m} \mathbb{Z}$$

Moreover, $k_i(1) = e_i$. (inclusion on the ith summand), and since the upper triangle commutes, we have $p_i(e_j) = 1$, $\forall j$. (i.e. p_i is the projection onto the ith summand)

By the commutativity of the lower triangle, we have $(p_i \circ j)(1) = 1$, and so it follows that:

$j(1) = (1, \dots, 1) = \Sigma_i k_i(1)$

Now the commutativity of upper square implies that $f_*(k_i(1))$

$$= \deg f|x_i \Rightarrow f_*(j(1)) = f_*(\Sigma_i k_i(1))$$
$$= \sum_i \deg(f|x_i)$$

Finally, the commutativity of the lower square implies that
$\deg(f) = \sum_i \deg(f|x_i)$. ∎

Examples

(a) Consider the maps
$S^n \xrightarrow{q} \vee_k S^n \xrightarrow{P} S^n$, where q collapses the complement of k disjoint balls B_i in S^n to a point and P identify

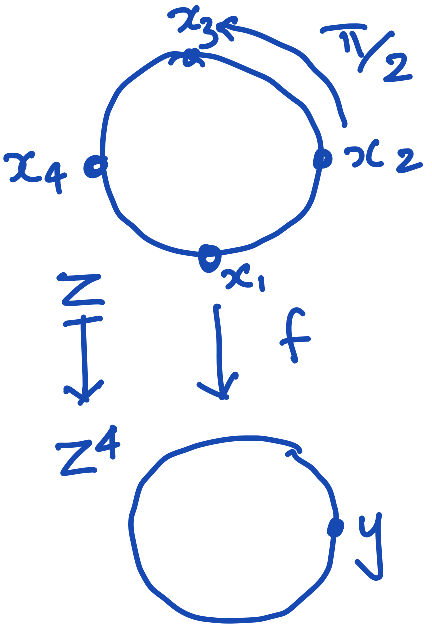
each of the resultant sphere summands to a single sphere.

Let $f = p \circ q$; then for almost all $y \in S^n$, we have $f^{-1}(y_i) = \{x_1, \dots x_k\}$ where $x_i \in B_i$.

Since f is a local homeomorphism at each x_i, we have $\deg(f|x_i) = \pm 1$.

By precomposing p with reflection of the summands of $V_k(S^n)$, we can produce maps $S^n \longrightarrow S^n$ of degree $\pm k$.

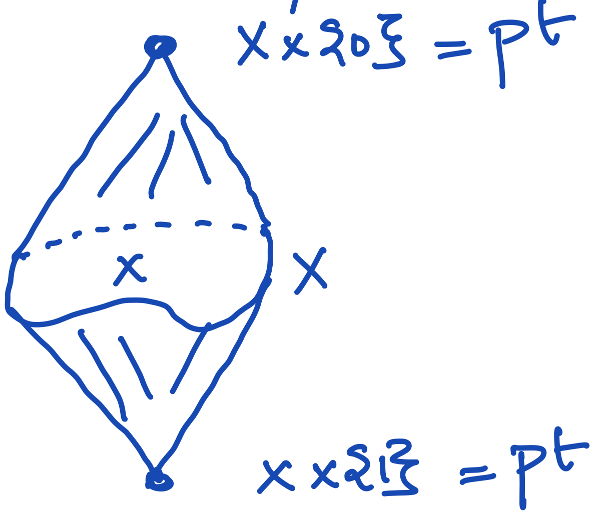
Example Consider the map $f : S^1 \longrightarrow S^1 : z \longmapsto z^k$. When k>0, f is a covering map and so we have $f^{-1}(y) = \{x_1, \dots x_k\}$ with f being a local homeo around each x_i.



A rotation has degree +1 as it is homotopic to id_{S^n}. Since around each point x_i, f can be homotoped to be restriction of a rotation, we have $\deg(f) = \sum_{i=1}^{K} \deg(f|_{x_i}) = k$.

Defn. The suspension SX of a space X is defined by

$$SX = X\times[0,1]/(X\times\{0\})\sqcup(X\times\{1\})$$



Thus, a map $f: X \longrightarrow Y$ suspends to a map $Sf: SX \longrightarrow SY$.

Proposition. For a map $f: S^n \to S^n$

$$\deg(f) = \deg(Sf).$$

Proof. First, we note $SS^n \approx S^{n+1}$.

Moreover, $CS^n = S^n \times I / S \times \{1\} (\cong D^{n+1})$, (the cone of S^n) has base $S^n \times \{0\}$, so $CS^n / S^n \cong S^{n+1} (= SS^n)$

Thus, the map f induces a $Cf : (CS^n, S^n) \longrightarrow (CS^n, S^n)$ with quotient $Sf : S^{n+1} (= CS^n/S^n) \to S^{n+1} (= CS^n/S^n)$

Thus, by naturality of the boundary maps in the LES of the pair (CS^n, S^n), we have the commutative diagram:

$$\begin{tikzcd}
\widetilde{H}_{n+1}(S^{n+1}) \arrow[r, "\bar{\partial}"] \arrow[d, "Sf_*"] & \widetilde{H}_n(S^n) \arrow[d, "f_*"] \\
\widetilde{H}_{n+1}(S^{n+1}) \arrow[r, "\bar{\partial}"] & \widetilde{H}_n(S^n)
\end{tikzcd}$$

Hence, $\deg(f) = \deg(Sf)$ $\blacksquare$

Cellular Homology

If X is CW-complex, then:

(a) $$H_k(X^n, X^{n-1}) = \begin{cases} 0, & \text{if } k \neq n \\ \langle \{n\text{-cells}\} \rangle, & \text{if } k = n \end{cases}$$

(b) $H_k(X^n) = 0$, for $k > n$. In particular, if X is finite-dimensional, then $H_k(X) = 0$, for $k > \dim(X)$.

(c) The inclusion $i : X^n \hookrightarrow X$ induces an isomorphism $i_* : H_k(X^n) \longrightarrow H_k(X)$, for $k < n$.

Proof (a) Since (X^n, X^{n-1}) is a good pair and $(X^n/X^{n-1}) \approx \bigvee_{i=1}^{|\{n\text{-cells}\}|} S^n$, we have:

$$H_K(X^n, X^{n-1}) \cong H_K\left(\bigvee_{i=1}^{|\{n\text{-cells}\}|} S^n\right)$$

$$\cong \begin{cases} \bigoplus_{i=1}^{|\{n\text{-cells}\}|} \mathbb{Z}, & \text{if } K = n \\ 0, & \text{if } K \neq n \end{cases}$$

(b) From the LES of the pair (X^n, X^{n-1}), we have:

$$\to H_{K+1}(X^n, X^{n-1}) \to H_K(X^{n-1}) \to H_K(X^n) \to H_K(X^n, X^{n-1}) \to \cdots$$

Here, $H_K(X^n, X^{n-1}) = 0$, for $K \neq n, n-1$.

So, $H_K(X^{n-1}) \cong H_K(X^n)$, for $K \neq n, n-1$.

Thus, for $k > n$, we have $H_k(X^n) \cong H_k(X^0) = 0$, as required.

(c) If $k < n$, then $H_k(X^n) \cong H_k(X^{n+m})$, for all $m \geq 0$, proving (c) if X is finite-dimensional.

The proof for the infinite-dimensional case is left as an exercise ∎

For a CW-complex, by Lemma above, we have the following diagram:

$$\begin{tikzcd}
 & & & & 0 \\
 & 0 \arrow[dr] & & H_n(X^{n-1}) \cong H_n(X) \arrow[ur] & & \\
 & & H_n(X^n) \arrow[ur] \arrow[dr, "j_n"] & & & \\
\cdots \arrow[r] & H_{n+1}(X^{n+1}, X^n) \arrow[ur, "\partial_{n+1}"] \arrow[rr, "d_{n+1}"] & & H_n(X^n, X^{n-1}) \arrow[rr, "d_n"] \arrow[dr, "\partial_n"'] & & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[r] & \cdots \\
 & & & & H_{n-1}(X^{n-1}) \arrow[ur, "j_{n-1}"'] & & \\
 & & & 0 \arrow[ur] & & &
\end{tikzcd}$$

Here, $d_{n+1} = j_n \circ \partial_{n+1}$ and $d_n = j_{n-1} \circ \partial_n$, and so $d_n \circ d_{n+1} = j_{n-1} \circ (\overset{0}{\overset{\shortparallel}{\partial_n \circ j_n}}) \circ \partial_{n+1}$

$$= 0$$

Thus, the horizontal row forms a chain complex, called the

cellular chain complex.

The homology groups of this chain complex are called the cellular homology groups $H_n^{CW}(X)$.

Theorem. $H_n^{CW}(X) \cong H_n(X)$.

Proof From the diagram above, it follows that:

$$H_n(X) \cong H_n(X^n) / \mathrm{Im}(\partial_{n+1}).$$

Since j_n is injective, we have:

(a) $\mathrm{Im}(j_n) = \ker(\partial_n)$.

(b) $\mathrm{Im}(\partial_{n+1}) \cong \mathrm{Im}(j_n \circ \partial_{n+1})$
$= \mathrm{Im}(d_{n+1})$

Since j_{n-1} is injective, we have (c) $\operatorname{Ker}(\partial_n) = \operatorname{Ker}(d_n)$. Thus from (a) - (c), it follows that j_n induces $\overline{j}_n : H_n(X^n)/\operatorname{Im}(\partial_{n+1}) \to \operatorname{Ker}(d_n)/\operatorname{Im}(d_{n+1}) = H_n^{CW}(X)$ $\blacksquare$

Corollary.

(a) If X is a CW-complex with k n-cells, then $H_n(X)$ is generated by at most k elements. In particular, if X no k-cells, then $H_n^{CW}(X) = 0$

(b) If X is a CW-complex with no two of its cells in adjacent dimensions, then $H_n(X) = \langle \{n\text{-cells in } X\} \rangle$.

Example. $\mathbb{CP}^n = \mathbb{C}^{n+1} - \{0\} / x \sim \lambda v$, for $\lambda \neq 0$. Equivalently, $\mathbb{CP}^n = S^{2n+1} (\subset \mathbb{C}^{n+1}) / v \sim \lambda v$, for $|\lambda| = 1$.

Claim. $\mathbb{CP}^n = D^{2n} / v \sim \lambda v$, for $v \in \partial D^{2n}$

Proof. The vectors in $S^{2n+1} \subset \mathbb{C}^{n+1}$ with last coordinate real and nonnegative are precisely vectors of the form $(w, \sqrt{1 - \|w\|^2}) \in \mathbb{C}^n \times \mathbb{C}$ with $\|w\| \leq 1$. These vectors form the graph of the function $w \xmapsto{f} \sqrt{1-\|w\|^2}$. Note that $Im(f)$ is a disk D^{2n}_+ bounded by the sphere $S^{2n-1} \subset S^{2n+1}$, where $S^{2n-1} = \{(w,0) \in \mathbb{C}^n \times \mathbb{C} \mid \|w\| = 1\}$.

Since each vectors in S^{2n+1} is equivalent under the identification $v \sim \lambda v$ to a unique D^{2n}_+ (if the last coordinate is zero), we have $v \sim \lambda v$, for $v \in S^{2n-1}$.

From this description, we see that $\mathbb{C}P^n = \mathbb{C}P^{n-1} \sqcup e^{2n}$, where e^{2n} is attached by the quotient map $S^{2n-1} \longrightarrow \mathbb{C}P^{n-1}$.

Thus, by induction, we have $\mathbb{C}P^n = e^0 \cup e^2 \cup \dots \cup e^{2n}$.

Therefore,

$$H_i(\mathbb{C}P^n) = \begin{cases} \mathbb{Z}, & i = 0, 2, 4, \dots, 2n \\ 0, & \text{otherwise.} \end{cases}$$

Proposition (cellular boundary formula).

$d_n(e^n_\alpha) = \sum_\beta d_{\alpha\beta} e^{n-1}_\beta$, where

$d_{\alpha\beta} = \deg(S^{n-1}_\alpha \longrightarrow X^{n-1} \longrightarrow S_\beta^{n-1})$

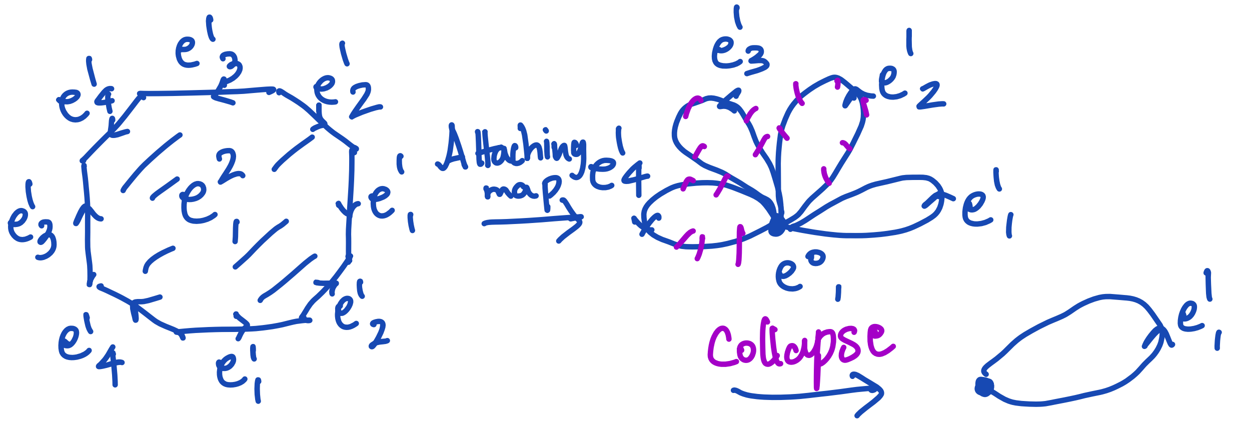
that is the composition of the attaching map of e^n_α with the quotient map collapsing $X^{n-1} - e_\beta^{n-1}$ to a point.

Examples (a) M_g has one 0-cell, 2g 1-cells $a_1, b_1, \dots a_g, b_g$ attached along $[a_1, b_1] \dots [a_g, b_g]$. The associated cellular chain complex is:

$$0 \to \mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^{2g} \xrightarrow{d_1} \mathbb{Z} \longrightarrow 0$$

As there is only one 0-cell, $d_1 = 0$.

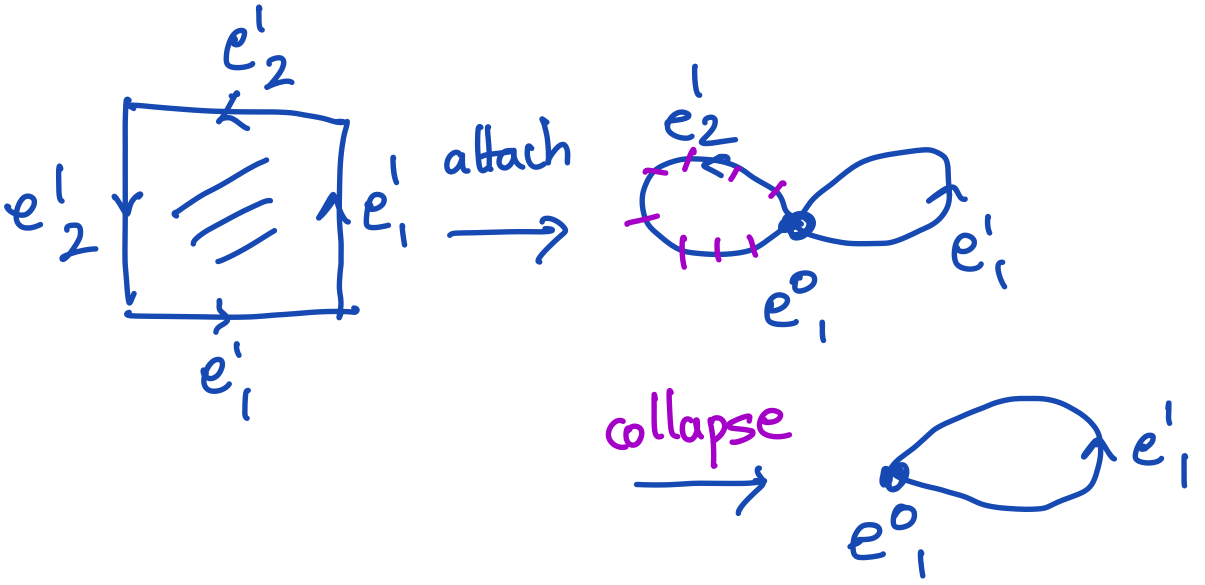
$$d_2(e^2_1) = \sum_{i=1}^{2g} d_{1i} e^{n-1}_i ,$$ where the one skeleton comprises the edges $\{e^1_1, e^1_2, \dots, e^1_{2g}\}$.



Thus, $d_2(e^2_1) = \sum_{i=1}^{2g} e^1_i - \sum_{i=1}^{2g} e^1_i = 0$

$$\Rightarrow \quad H_n(M_g) = \begin{cases} \mathbb{Z}^{2g}, & \text{if } n=1 \\ \mathbb{Z}, & \text{if } n=0,2 \\ 0, & \text{otherwise.} \end{cases}$$

(b) Non orientable surface N_g of genus g.



$$0 \longrightarrow \mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^g \xrightarrow{d_1} \mathbb{Z} \longrightarrow 0$$

As in the case of M_g, $d_1 = 0$.

Moreover, $d_2(e_1^2) = 2\sum_{i=1}^{g} e_i^1$

$$= 2(e_1^1 + \cdots + e_g^1),$$

i.e. $d_2(1) = (2, \cdots 2)$

$$H_1(N_g) = \frac{\operatorname{Ker}(d_1)}{\operatorname{Im}(d_2)} \cong \frac{\mathbb{Z}^g}{\langle (2, \cdots; 2) \rangle}$$

$$\cong \frac{\langle e_1, e_2, \ldots, e_{g-1}, e_1 + \cdots + e_g \rangle}{\langle 2(e_1 + \cdots + e_g) \rangle}$$

$$\cong \mathbb{Z}^{g-1} \oplus \mathbb{Z}_2$$

$$H_n(N_g) = \begin{cases} \mathbb{Z}, & n = 0, 2 \\ \mathbb{Z}^{2g-1} \oplus \mathbb{Z}_2, & n = 1 \\ 0, & \text{otherwise.} \end{cases}$$

(c) $\mathbb{R}P^n$ has a CW-structure with one cell e^k in each dimens'n $k \le n$ and e^k attached via 2-sheeted $\varphi : S^{k-1} \to \mathbb{R}P^{k-1}$.

$$0 \xrightarrow{d_n} \mathbb{Z} \xrightarrow{d_{n-1}} \mathbb{Z} \to \cdots \to \mathbb{Z} \xrightarrow{d_1} \mathbb{Z} \xrightarrow{d_0} 0$$

$$d_k = \deg\left(S^{k-1} \xrightarrow{\varphi} \mathbb{R}P^{k-1} \xrightarrow{q} \overset{\overset{S^{k-1}}{\|}}{\mathbb{R}P^{k-1}/\mathbb{R}P^{k-2}}\right)$$

Note that $S^{k-1} \setminus S^{k-2} = D_2^1 \sqcup D_2^2$

and $(q \circ \varphi)|D_2^{\overset{\circ}{i}} = h_i$ is a homeo

such that $h_2 = h_1 \circ a$

Thus, we have $\deg(q \circ \varphi)$

$= \deg(id) + \deg(a)$

$= 1 + (-1)^k$.

So, $d_k = \begin{cases} 0, & \text{if } k \text{ is odd} \\ 2, & \text{if } k \text{ is even} \end{cases}$

$$H_k(\mathbb{R}P^n) = \begin{cases} \mathbb{Z}, \text{ if } k=0 \text{ \& } k=n \text{ odd} \\ \mathbb{Z}_2, \text{ if } k \text{ odd } 0<k<n \\ 0, \text{ otherwise}. \end{cases}$$

Euler Characteristic

For a finite CW-complex X, the Euler characteristic $\chi(X)$ is defined to be $\sum_n (-1)^n c_n$, where c_n is the number of n-cells of X.

Theorem $\chi(X) = \sum_n (-1)^n \operatorname{rank} H_n(X)$

Proof. Here rank is the number of free generators of $H_n(X)$.

For a short exact sequence of finitely generated abelian groups $0 \to A \to B \to C \to 0$, we have $\mathrm{rank}(B) = \mathrm{rank}(A) \oplus \mathrm{rank}(C)$.

Now, we consider the chain complex:

$$\to C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \to \cdots \to C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} 0,$$

where $C_n = H_n(X^n, X^{n-1})$.

This leads to two SESs:

(a) $0 \to \mathrm{Ker}(d_n) \to C_n(X) \xrightarrow{d_n} \mathrm{Im}(d_n) \to 0$

(b) $0 \to \mathrm{Im}(d_{n+1}) \to \mathrm{Ker}(d_n) \to H_n(X) \to 0$

From ⓐ and ⓑ, we have:

(i) $\mathrm{Rank}(C_n(X)) = \mathrm{Rank}(\mathrm{Im}(d_n)) + \mathrm{Rank}(\mathrm{Ker}(d_n))$

(ii) $\mathrm{Rank}(\mathrm{Ker}(d_n)) = \mathrm{Rank}(\mathrm{Im}(d_{n+1})) + \mathrm{Rank}(H_n(X))$

Sub (ii) in (i), multiplying by $(-1)^n$ and summing over n, we get:

$$\sum_n (-1)^n \text{Rank}(C_n) = \sum_n (-1)^n \Big(\text{Rank}(\text{Im}(d_n)) + \text{Rank}(\text{Im}(d_{n+1}))\Big) + \sum_n (-1)^n \text{Rank}(H_n(X))$$

$$\rightarrow \sum_n (-1)^n \text{Rank}(C_n) = \sum_n (-1)^n \text{Rank}(H_n) \quad \blacksquare$$

Splitting Lemma. For a SES $0 \rightarrow A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0$ of groups the following statements are equivalent:

(a) $B \cong A \times C$

(b) $\exists$ a homomorphism $p : B \to A$ such that $p \circ i = id_A$

(c) $\exists$ a homomorphism $q : C \to B$ such that $j \circ q = id_C$.

In particular, if A, B and C are abelian, then the statement in (a) takes the form
$A \cong B \oplus C$.

Proposition. If $r : X \longrightarrow A$ is a retraction onto a subspace, then $H_n(X) \cong H_n(A) \oplus H_n(X, A)$

Proof. If $i : A \hookrightarrow X$ is the inclusion; then $r \circ i = id_A$ $\Rightarrow$ $r_* \circ i_* = \left(id_{H_n(A)}\right)$. Thus, the SES

$$\begin{tikzcd} 0 \arrow[r] & H_n(A) \arrow[r, "i_*"] & H_n(X) \arrow[r, "j_*"] \arrow[l, bend left, "r_*"] & H_n(X,A) \arrow[r] & 0 \end{tikzcd}$$

splits, yielding the assertion ■

Examples

(a) Suppose ∃ a retraction $r : D^n \longrightarrow S^{n-1}$. Then

$$H_{n-1}(D^n) \cong H_{n-1}(S^{n-1}) \oplus H_{n-1}(D^n, S^{n-1}),$$

which is impossible.

(b) Suppose that the mapping cylinder M_f of a map $f: S^n \to S^n$ of degree $m > 1$ retracted onto $S^n \subset M_f$, then $\exists$ a split SES

$$\begin{tikzcd}
0 \arrow[r] & H_n(S^n) \arrow[r] \arrow[d, phantom, sloped, "\cong"] & H_n(M_f) \arrow[r] & H_n(M_f, S^n) \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "m"] & \mathbb{Z} \arrow[r] & \mathbb{Z}_m \arrow[r] & 0
\end{tikzcd}$$

$\#$.

Mayer-Vietoris Sequence

Let $A, B \subset X$ such that $X = \mathring{A} \cup \mathring{B}$.

Let $C_n(A+B)$ be the subgroup of $C_n(X)$ consisting of chains that are sums of chains in A and B.

Then $\partial : C_n(X) \longrightarrow C_{n-1}(X)$ takes $C_n(A+B) \xrightarrow{\partial} C_{n-1}(A+B)$. So $\exists$ a chain complex of $A+B$.

Moreover, $C_n(A+B) \hookrightarrow C_n(X)$ induce isomorphism on homology groups. (Proof of Excision)

Thus, the SES of chain complexes

$$0 \to C_n(A \cap B) \xrightarrow{\varphi} C_n(A) \oplus C_n(B) \xrightarrow{\psi} C_n(A+B) \to 0$$

where $\varphi(x) = (x, -x)$ and $\psi(x, y) = x+y$ yields a LES of homology groups called the Mayer-Vietoris sequence.

Theorem. $\exists$ a LES of homology groups given by:

$$\cdots \to H_n(A\cap B) \xrightarrow{\Phi} H_n(A)\oplus H_n(B) \xrightarrow{\Psi} H_n(X) \xrightarrow{\partial} H_{n-1}(A\cap B), \cdots$$

where Φ is induced by $\varphi : C_n(A\cap B) \longrightarrow C_n(A)\oplus C_n(B)$ given by $\varphi(x) = (x, -x)$ and Ψ is induced by $\psi : C_n(A)\oplus C_n(B) \longrightarrow C_n(X)$.

Examples (a) Take $X = S^n = A\cup B$, where A and B are northern and southern hemispheres with $A\cap B = S^{n-1}$.

Then the reduced M-V sequence yields:

$$\to \overset{\overset{0}{\|}}{\tilde{H}_i(A)\oplus\tilde{H}_i(B)} \to \tilde{H}_i(S^n)$$

$$\to \tilde{H}_{i-1}(S^{n-1})$$

$$\to \overset{\overset{0}{\|}}{\tilde{H}_{i-1}(A)+\tilde{H}_{i-1}(B)}$$

$$\to \cdots$$

$$\Rightarrow \tilde{H}_i(S^n)\cong\tilde{H}_{i-1}(S^{n-1})$$

(b) The Klein bottle $K = A\cup B$, where A, B are Möbius bands glued along their boundary circles. By the M-V sequence, we have

$$0 \to H_2(K) \xrightarrow{\partial} H_1(A\cap B) \xrightarrow{\Phi} H_1(A)\oplus H_1(B)$$

$$\xrightarrow{\Psi} H_1(K) \longrightarrow 0$$

$$0 \to H_2(K) \xrightarrow{\partial} \mathbb{Z} \xrightarrow{1\mapsto(2,-2)} \mathbb{Z}\oplus\mathbb{Z}$$

$$\to H_1(K) \to 0$$

$$H_2(K) \cong \mathrm{Im}(\partial) = \mathrm{Ker}(\Phi) = \{0\}$$

$$H_1(A) \oplus H_1(B) / \mathrm{Ker}(\Psi) \cong H_1(K)$$

$$H_1(A) \oplus H_1(B) / \mathrm{Im}(\Phi) \cong H_1(K)$$

$$\cong \frac{\langle (1,0), (1,-1) \rangle}{\langle (2,-2) \rangle} \cong \mathbb{Z} \oplus \mathbb{Z}_2$$

Cohomology

Let X be space and G an abelian group. Consider the chain complex of free abelian group

$$\cdots \longrightarrow C_{n+1}(X) \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \rightarrow \cdots$$

We dualize this complex by considering the cochain groups

$$C_n^*(X) = \text{Hom}(C_n(X), G), \ \forall n.$$

Then for each n, ∂_n induces a map:

$$C_n^*(X) \xleftarrow{\delta_n} C_{n-1}^*(X)$$

Since $\partial_n \circ \partial_{n+1} = 0$, it follows that

$$\delta_{n+1} \circ \delta_n = 0$$

Thus, we obtain a <u>dual chain complex</u> :

$$\cdots \to C^*_{n+1}(X) \xleftarrow{\delta_{n+1}} C^*_n(X) \xleftarrow{\delta_n} C^*_{n-1}(X) \to \cdots ;$$

an we define the n^{th} Cohomology group by :

$$H^n(X;G) = \frac{Ker(\delta_{n+1})}{Im(\delta_n)} .$$

<u>Theorem</u> (Universal Coefficient Theorem). For each n, $\exists$ a split SES given by

$$0 \to Ext(H_{n-1}(X), G) \to H^n(X;G) \xrightarrow{h} Hom(H_n(X), G) \to 0$$

Lemma 1. $\exists$ a natural hom $h : H^n(X;G) \longrightarrow \mathrm{Hom}(H_n(X);G)$.

Proof A cohomology class $[\varphi] \in H^n(X;G)$ is represented by a hom $\varphi : C_n(X) \longrightarrow G$ such that $\delta_{n+1}(\varphi) = 0$

$\Longrightarrow \varphi \circ \partial_{n+1} = 0 \Longrightarrow \varphi$ vanishes on $\mathrm{Im}(\partial_{n+1})$. Thus, $\overset{\varphi_0}{\overset{\shortparallel}{\varphi|\mathrm{Ker}(\partial_n)}}$ induces a $\overline{\varphi_0} : H_n(X) \longrightarrow G$.

Moreover, as $\varphi \in \mathrm{Im}(\delta_n)$, we have

$\varphi = \delta_n(\psi) = \psi(\partial_n)$, and so it follows $\overline{\varphi_0} = 0$ in $\mathrm{Ker}(\partial_n)$.

Thus, my mapping $\varphi \overset{h}{\longmapsto} \overline{\varphi_0}$. we get a well-defined hom. ∎

Lemma 2. h is surjective.

Proof. Consider the SES

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}(\partial_n) \arrow[r] & C_n(X) \arrow[r, "\partial_n"] \arrow[l, dotted, bend left, "p"] & \mathrm{Im}(\partial_n) \arrow[r] & 0
\end{tikzcd}$$

Note that this splits since $\mathrm{Im}(\partial_n)$ is a free subgroup of $C_{n-1}(X)$.

Thus, $\exists$ a $p : C_n(X) \to \mathrm{Ker}(\partial_n)$ such that $p|_{\mathrm{Ker}(\partial_n)} = \mathrm{Id}_{\mathrm{Ker}(\partial_n)}$.

Composing $\varphi_0 : \mathrm{Ker}(\partial_n) \longrightarrow G$ with p, we obtain an extension of $\varphi_0 = \varphi|_{\mathrm{Ker}(\partial_n)}$ to $\varphi = \varphi_0 \circ p : C_n(X) \longrightarrow G$.

Thus, this extends homs $\mathrm{Ker}(\partial n) \to G$ that vanish in $\mathrm{Im}(\partial n{+}1)$ to homs $C_n(X) \to G$ that vanish in $\mathrm{Im}(\partial n{+}1)$.

In other words, we obtain a hom.

$$\mathrm{Hom}(H_n(X);G) \longrightarrow \mathrm{Ker}(\delta n{+}1)$$

Composing this with the quotient map

$$\mathrm{Ker}(\delta n{+}1) \longrightarrow H^n(X;G)$$

yields a hom

$$\mathrm{Hom}(H_n(X),G) \xrightarrow{\alpha} H^n(X;G)$$

such that $h \circ \alpha = \mathrm{id}_{\mathrm{Hom}(H_n(X);G)}$

$\Longrightarrow$ h is surjective and we obtain a split SES.

$$0 \longrightarrow \mathrm{Ker}(h) \longrightarrow H^n(X;G) \xrightarrow{h} \mathrm{Hom}(H_n(X);G) \longrightarrow 0$$

<u>Defn</u>. A <u>free resolution</u> of an abelian group H is an exact sequence

$$\cdots \longrightarrow F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \longrightarrow 0,$$

where each F_n is free.

<u>Lemma</u>. Given free resolutions F and F' of abelian groups H and H', every hom $\alpha : H \to H'$ can be extended to a chain map F to F':

$$\begin{tikzcd}
\cdots \arrow[r] & F_2 \arrow[r, "f_2"] \arrow[d, "\alpha_2"] & F_1 \arrow[r, "f_1"] \arrow[d, "\alpha_1"] & F_0 \arrow[r, "f_0"] \arrow[d, "\alpha_0"] & H \arrow[r] \arrow[d, "\alpha"] & 0 \\
\cdots \arrow[r] & F_2' \arrow[r, "f_2'"'] & F_1' \arrow[r, "f_1'"'] & F_0' \arrow[r, "f_0'"'] & H' \arrow[r] & 0
\end{tikzcd}$$

Furthermore, any two such chain maps extending α are chain homotopic.

(b) For any two free resolutions F and F' of H, $\exists$ canonical isomorphisms $H^n(F;G) \cong H^n(F';G)$ for all n.

Example Every abelian group has a free resolution of the form

$$0 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow H \longrightarrow 0$$

with $F_i = 0$ for $i > 1$.

Take F_0 to be free abelian group with basis bijective with a chosen gen. set for H. Then $\exists$ a natural hom $f_0 : F_0 \longrightarrow H$ sending basis elts $\longrightarrow$ chosen generators. Setting $F_1 = \text{Ker}(f_0)$, we obtain the required free resolution. F.

Note that $H^n(F;G) = 0$ for $n > 1$.

Defn. We define

$$\text{Ext}(H;G) := H^1(F;G).$$

Lemma 3. $\mathrm{Ker}(h) \cong \mathrm{Ext}(H_{n-1}(X);G)$

Proof. Considering the dual of

$$\begin{tikzcd}
0 \arrow[r] & Z_{n+1} \arrow[r] \arrow[d, "0"] & C_{n+1} \arrow[r, "\partial"] \arrow[d, "\partial"] & B_n \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & Z_n \arrow[r] & C_n \arrow[r] & B_{n-1} \arrow[r] & 0
\end{tikzcd}$$

yields the following diagram:

$$(*) \quad \begin{tikzcd}
0 & Z_{n+1}^* \arrow[l] & C_{n+1}^* \arrow[l] & B_n^* \arrow[l] & 0 \arrow[l] \\
 & & \delta(\phi) & \phi_0|B_n \arrow[l] & \\
 & \phi_0 & \phi \arrow[l] \arrow[u] & & \\
0 & Z_n^* \arrow[l] \arrow[uuu, "0"] & C_n^* \arrow[l] \arrow[uuu, "\delta"] & B_{n-1}^* \arrow[l] \arrow[uuu, "0"] & 0 \arrow[l]
\end{tikzcd}$$

The rows of $(*)$ are also exact. (dual of a split SES is a split SES)

has an associated LES

$$(**) \quad \cdots \leftarrow B_n^* \xleftarrow{i_n^*} Z_n^* \leftarrow H^n(X;G) \leftarrow B_{n-1}^* \xleftarrow{i_{n-1}^*} Z_{n-1}^* \leftarrow \cdots$$

Here the "boundary map" i_n^* is the dual of the inclusion $i_n : B_n \longrightarrow Z_n$. This is consistent with the manner in which such maps are defined traditionally (via diagram chasing).

Note that $i_n^*(\phi) = \phi|_{B_n}$.

The LES breaks into SESs

$$0 \leftarrow \mathrm{Ker}(i_n^*) \leftarrow H^n(X,G) \longleftarrow \mathrm{Coker}(i_{n-1}^*) \longleftarrow 0$$

Since
$$\mathrm{Ker}(i_n^*) = \{ Z_n \xrightarrow{\varphi} G \mid \varphi|_{B_n} = 0 \}$$
$$= \{ Z_n/B_n \longrightarrow G \}$$
$$= \mathrm{Hom}(H_n(X), G)$$

Also, note that the map $H^n(X;G) \to \ker(i_n^*)$ becomes h.

Thus, by Lemma 2, we have a split SES

$$0 \to \text{Coker}(i_{n-1}^*) \longrightarrow H^n(X;G) \longrightarrow \text{Hom}(H_n(X),G) \to 0$$

Finally, it follows the Lemma on free resolutions that

$$\text{Coker}(i_{n-1}^*) = \text{Ext}(H_{n-1}(X);G)$$

by considering the free resolution

$$0 \longrightarrow B_{n-1} \xrightarrow{i_{n-1}} Z_{n-1} \longrightarrow H_{n-1}(C) \to 0$$

∎

Proposition.

(a) $\mathrm{Ext}(H \oplus H', G) \cong \mathrm{Ext}(H, G) \oplus \mathrm{Ext}(H, G)$

(b) $\mathrm{Ext}(H, G) = 0$, if H is free

(c) $\mathrm{Ext}(\mathbb{Z}_n, G) \cong G/nG$

Proof

(a) Follows from the fact that the direct sum of free resolutions is a free resolution.

(b) When H is free, the free resolution $0 \longrightarrow H \longrightarrow H \to 0$ yields the assertion.

(c) Consider the dual of the free resolution

$$0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \longrightarrow \mathbb{Z}_n \longrightarrow 0,$$

This yields the exact sequence

$$\begin{tikzcd} & & G \arrow[d, phantom, "\cong" rotate=90] & G \arrow[d, phantom, "\cong" rotate=90] & & \\ 0 & \mathrm{Ext}(\mathbb{Z}_n, G) \arrow[l] & \mathbb{Z}^* \arrow[l] & \mathbb{Z}^* \arrow[l, "n"'] & \mathbb{Z}_n^* \arrow[l] & 0 \arrow[l] \end{tikzcd}$$

and the assertion follows ■.

<u>Corollary</u>. If $H_n(X)$ is finitely generated for all n with torsion component T_n, then:

$$H^n(X;\mathbb{Z}) \cong \frac{H_n(X)}{T_n} \oplus T_{n-1}$$

<u>Corollary</u>. If a chain map between chain complexes induces

isomorphisms on homology groups, then it induces isomorphisms on cohomology groups.

Remark. The algebraic machinery of UCT can be generalized to modules over a ring R by considering R-modulus homs (Hom_R) instead of Hom. This will use the fact that submodules of free R-modules are free if R is a PID.

Reduced cohomology. By dualizing the augmented chain complex $\cdots \longrightarrow C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$. By applying UCT, we see that:

$$\tilde{H}^n(X;G) = \begin{cases} H^n(X;G), & \text{for } n>0 \\ \mathrm{Hom}(H_0(X), G), & \text{for } n=0. \end{cases}$$

LES of pair. We dualize the SES $0 \to C_n(A) \to C_n(X) \longrightarrow C_n(X,A) \longrightarrow 0$ to obtain LES of cohomology groups:

$$\cdots \to H^n(X,A;G) \xrightarrow{j^*} H^n(X;G) \xrightarrow{i^*} H^n(A;G) \xrightarrow{\delta} H^{n+1}(X,A;G) \to \cdots$$

An analogous sequence holds for triples.

Induced homs. By dualizing the chain maps $f_\#: C_n(X) \to C_n(Y)$ we get the cochain maps $f^\#: C^n(X) \to C^n(Y)$. The relation $f_\#\partial = \partial f_\#$ dualizes to $\delta f^\# = f^\# \delta$, so $f^\#$ induces

$$f^*: H^n(Y;G) \longrightarrow H^n(X;G).$$

The same reasoning also holds for maps $f : (X, A) \longrightarrow (Y, B)$

Homotopy invariance. If $f \simeq g : (X, A) \longrightarrow (Y, B)$, then $g_\# - f_\# = \partial P + P\partial$ dualizes to $g^\# - f^\# = P^*\delta + \delta P^*$. Thus, we have $f^* = g^*$.

Excision. As in the case of homology, for subspaces $Z \subset A \subset X$ with $\overline{Z} \subset A^\circ$, the inclusion $i : (X-Z, A-Z) \hookrightarrow (X, A)$ induces

$i^* : H^n(X,A;G) \longrightarrow H^n(X-Z, A-Z; G)$

for all n.

Mayer-Vietoris. If $X = A^\circ \cup B^\circ$,

$\exists$ a LES

$$\cdots \longrightarrow H^n(X;G) \xrightarrow{\Psi} H^n(A;G) \oplus H^n(B;G)$$

$$\xrightarrow{\Phi} H^n(A \cap B;G)$$

$$\longrightarrow H^{n+1}(X;G) \longrightarrow \cdots$$

Cup Product

Let $R = \mathbb{Z}, \mathbb{Z}_n$ or $\mathbb{Q}$. For cochains $\varphi \in C^k(X;R)$ and $\psi \in C^l(X;R)$, the cup product $\varphi \cup \psi \in C^{k+l}(X;R)$ is the cochain whose value on a singular simplex $\sigma : \Delta^{k+l} \longrightarrow X$ is given by the formula.

$$(\varphi \cup \psi)(\sigma) = \varphi(\sigma|[v_0, \dots, v_k])\,(\psi(\sigma|[v_k, \dots v_{k+l}])$$

Here the RHS is a product in R.

<u>Lemma</u>. $\delta(\varphi \cup \psi) = \delta\varphi \cup \psi + (-1)^k \varphi \cup \delta\psi$ for $\varphi \in C^k(X;R)$ and $\psi \in C^l(X;R)$.

<u>Proof</u> For $\sigma : \Delta^{k+l+1} \longrightarrow X$, we have:

$$(\delta\varphi \cup \psi)(\sigma) = \sum_{i=0}^{k+1} (-1)^i \varphi(\sigma|[v_0, \dots, \hat{v_i}, \dots v_{k+1}])\,\psi(\sigma|[v_{k+1}, \dots, v_{k+l+1}])$$

$$(-1)^K(\varphi \cup \delta\psi)(\sigma) = \sum_{i=K}^{K+l+1} (-1)^i \varphi(\sigma|[v_0,\dots,v_K])\,\psi(\sigma|[v_K,\dots,\hat{v}_i,\dots v_{K+l+1}])$$

The last term of first sum cancels in the first term of the second sum.

Since $\partial\sigma = \sum_{i=0}^{k+l+1} (-1)^i \sigma|[v_0,\dots,\hat{v}_i,\dots v_{k+l+1}]$, what remains is $\delta(\varphi\cup\psi)(\sigma) = (\varphi\cup\psi)(\partial\sigma)$ ■

Remark.

(a) From the lemma, it follows that the cup product of 2 cocycles is a cocycle.

(b) The cup product of a cocycle and a coboundary is a coboundary because

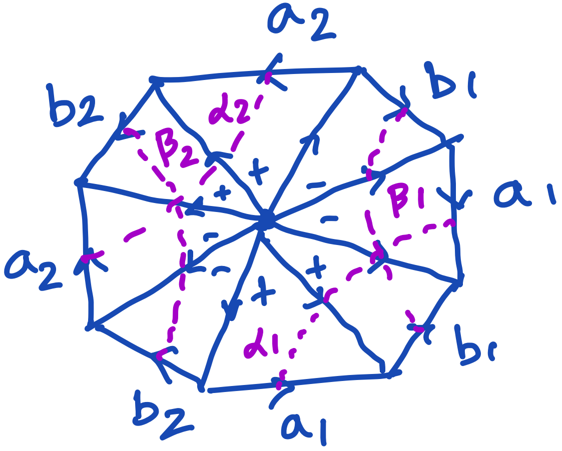
(i) $\varphi\cup\delta\psi = \pm\delta(\varphi\cup\psi)$, if $\delta\varphi = 0$

(ii) $\delta\varphi\cup\psi = \delta(\varphi\cup\psi)$, if $\delta\psi = 0$

Thus, there is an induced cup product map

$$H^k(X;R) \times H^\ell(X;R) \xrightarrow{\cup} H^{k+\ell}(X;R)$$

Example (a) Let M_g – closed orientable surface of genus $g \geq 1$



The cup product of interest is

$$H^1(M_2) \times H^1(M_2) \longrightarrow H^2(M_2)$$

By UCT, $H^1(M) \cong \text{Hom}(H_1(M), \mathbb{Z})$

Thus, a basis for $H_1(M_2)$ determines a dual basis for $\text{Hom}(H_1(M_2); \mathbb{Z})$.

In particular, a dual α_i of a_i assigns value 1 to a_i and 0 on the remaining basis elements.

Similarly, we have a dual β_i for b_i

Define a cocycle φ_i to have value one on the edges that meet arc α_i and zero elsewhere

Similarly, define ψ_i counting intersection with b_i

Then $\varphi_1 \cup \psi_1$ takes value 0 on all 2-simplices except the one with outer edge b_1 on the lower right on which it takes 1.

So $\varphi \cup \psi_1$ takes 1 on the 2-chain c formed by the sum of the 2-simplices with signs indicated.

Since $\partial c = 0$ and there are no 3-simplices c is not a boundary.

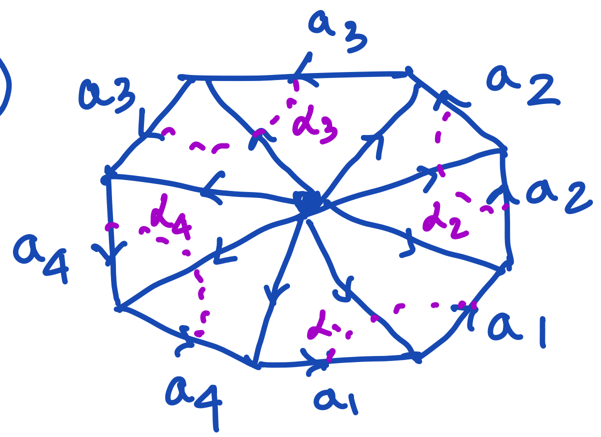
$\Rightarrow$ $[c]$ is a notrivial class in $H_2(M)$.

Since $(\varphi_1 \cup \psi_1)(c)$ generates $\mathbb{Z}$, it follows that $[c]$ is a generator $H_2(M_2) \cong \mathbb{Z}$ and $[\varphi_1 \cup \psi_1]$ generates $H^2(M_2) \cong \mathbb{Z}$.

In general,

$$\varphi_i \cup \psi_j = \begin{cases} \varphi_i \cup \psi_j \neq 0, & i=j \\ 0, & i \neq j \end{cases}$$

$$= -(\psi_j \cup \varphi_i)$$

(b) 

For the non-orientable surface, we use $\mathbb{Z}_2$-coefficients.

As before, for each a_i we choose the dual basis α_i $(H^1(N,\mathbb{Z}_2)$
$= \text{Hom}(H_1(N);\mathbb{Z}_2)$

As before, $\alpha_i \cup \alpha_j = \begin{cases} \neq 0 & \text{if } j=i \\ 0 & \text{if } i \neq j \end{cases}$.

<u>Proposition</u>. For a map $f: X \longrightarrow Y$, the induced maps $f^*: H^n(Y;R) \longrightarrow H^n(X;R)$ satisfy $f^*(\alpha \cup \beta) = f^*(\alpha) \cup f^*(\beta)$

Proof. It suffices to show that $f^{\#}(\varphi) \cup f^{*}(\psi) = f^{*}(\varphi \cup \psi)$.

$$\begin{aligned}(f^{\#}\varphi \cup f^{\#}\psi)(\sigma) &= (f^{\#}\varphi)(\sigma|[v_0,\dots,v_k]) \\ &\qquad (f^{\#}\psi)(\sigma|[v_k,\dots,v_{k+l}]) \\ &= \varphi(f\sigma|[v_0,\dots,v_k]) \\ &\qquad \psi(f\sigma|[v_k,\dots,v_{k+l}]) \\ &= (\varphi\cup\psi)(f\sigma) \\ &= f^{\#}(\varphi\cup\psi)(\sigma).\end{aligned}$$

■

The absolute and general forms are the maps:

$$H^k(X;R) \times H^l(Y,R) \xrightarrow{\times} H^{k+l}(X\times Y;R)$$

$$H^k(X,A;R) \times H^l(Y,B;R) \xrightarrow{\times} H^{k+l}(X\times Y, A\times Y \cup X\times B; R)$$

defined by $a \times b = p_1^*(a) \cup p_2^*(b)$, where p_1 & p_2 are the projections of $X \times Y$ onto X and Y.

Defn (Cohomology ring). The direct sum $\bigoplus_{n \geq 0} H^n(X;R) := H^*(X;R)$ comprises finite sums $\sum_i \alpha_i$ with $\alpha_i \in H^i(X;R)$, and the product of two such sums is defined to be $(\sum_i \alpha_i)(\sum_j \beta_j) = \sum_{i,j} \alpha_i \beta_j$. Thus, $H^*(X;R)$ is a ring (with identity) if R is a ring (with identity), called the <u>cohomology ring</u>.

Remarks. We may regard $H^*(X;R)$ as a graded ring i.e. a ring with decomposition as a sum $\bigoplus_{k\geq 0} A_k$ of additive subgroups A_k such that multiplication takes $A_k \times A_l$ to A_{k+l}

The simplest graded rings are polynomial rings $R[x]$ and their truncated verision $R[x]/(x^n)$ consisting of polynomials of degree $\leq n$.

Example. Let X be the 2-dimensional CW-complex obtained by attaching a 2-cell to S^1 by the degree m map $S^1 \longrightarrow S^1 : z \longmapsto z^m$.

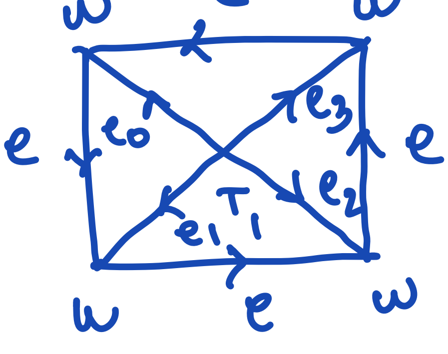
By UCT and cellular homology, we have:

$$H^n(X;\mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{for } n=0 \\ \mathbb{Z}_m, & \text{for } n=2 \end{cases}$$

$\Rightarrow$ cup product structure is unintersting

However, with $\mathbb{Z}_m$ coefficients

$$H^i(X;\mathbb{Z}_m) \cong \mathbb{Z}_m \quad \text{for } i=0,1,2.$$



A generator α of $H^1(X;\mathbb{Z}_m)$ is rep. by a cocycle φ assigning the value 1 to the edge e, which generates $H_1(X)$.

Since φ is a cocycle, we have

$$\varphi(e_i) + \varphi(e) = \varphi(e_{i+1}), \text{ for all } i.$$

So we may take $\varphi(e_i) = i \in \mathbb{Z}_m$, and hence:

$$(\varphi \cup \varphi)(T_i) = \varphi(e_i)\varphi(e) = i$$

Since $\sum_i T_i$ is a gen of $H_2(X;\mathbb{Z}_m)$ and there are 2-cocycles taking value 1 on $\sum_i T_i$, we have

$$h : H^2(X;\mathbb{Z}_m) \longrightarrow \mathrm{Hom}(H_2(X;\mathbb{Z}_m), \mathbb{Z}_m)$$

is an isomorphism.

The cocycle $\varphi \cup \varphi$ takes the value $\sum_{i=0}^{m-1} i$ on $\sum_i T_i$, and hence rep $\left(\sum_{i=0}^{m-1} i\right) \beta \in H^2(X; \mathbb{Z}_m)$, where β is a gen of $H^2(X; \mathbb{Z}_m)$.

In $\mathbb{Z}_m$,

$$\sum_{i=0}^{m-1} i \equiv \begin{cases} 0, & \text{if } m \text{ is odd} \\ k, & \text{if } m = 2k \end{cases}$$

Thus,

$$\alpha \cup \alpha = \alpha^2 = \begin{cases} 0, & \text{if } m \text{ is odd} \\ k\beta, & \text{if } m = 2k \end{cases}$$

In particular, $X = \mathbb{R}P^2$; $\alpha^2 = \beta$ in $H^2(\mathbb{R}P^2; \mathbb{Z}_2)$.

From these examples it follows that

$$H^*(\mathbb{R}P^2;\mathbb{Z}_2) = \frac{\mathbb{Z}_2(\alpha)}{(\alpha^3)}.$$

Theorem. $H^*(\mathbb{R}P^n;\mathbb{Z}_2) \cong \dfrac{\mathbb{Z}_2[\alpha]}{(\alpha^{n+1})}$

and $H^*(\mathbb{R}P^\infty;\mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha]$, where $|\alpha|=1$. In the complex case, $H^*(\mathbb{C}P^n;\mathbb{Z}) \cong \mathbb{Z}[\alpha]/(\alpha^{n+1})$ and $H^*(\mathbb{C}P^\infty;\mathbb{Z}) \cong \mathbb{Z}[\alpha]$, where $|\alpha|=2$.

Lemma. (a) The inclusions $i_\alpha : X_\alpha \hookrightarrow \sqcup_\alpha X_\alpha$ induces a ring isomorphism

$$H^*(\sqcup_\alpha X_\alpha ; R) \xrightarrow{\cong} \prod_\alpha H^*(X_\alpha ; R).$$

with respect to the usual coordinatewise multiplication in a product ring.

(b) Similarly, the inclusions $i_\alpha : X_\alpha \hookrightarrow \vee_\alpha X_\alpha$ induces a ring isomorphism

$$\prod_\alpha \tilde{H}^*(X_\alpha ; R) \xrightarrow{\cong} \tilde{H}^*(\vee_\alpha X_\alpha ; R)$$

Example. Consider the spaces $\mathbb{C}P^2$ and $S^2 \vee S^4$. Using homology or simply the additive structure of cohomology one cannot distinguish between these spaces. However, the cup product structure of these spaces are different.

To see this, note that the square of each element of $H^2(S^2 \vee S^4; \mathbb{Z})$ is zero since $\exists$ a ring isom.:

$$\tilde{H}^*(S^2 \vee S^4; \mathbb{Z}) \cong \tilde{H}^*(S^2; \mathbb{Z}) \oplus \tilde{H}^*(S^4; \mathbb{Z})$$

But the square of a generator of $H^2(\mathbb{C}P^2; \mathbb{Z})$ is nonzero by an earlier theorem.

<u>Theorem</u>. Given a commutative ring R, for all $\alpha \in H^k(X, A; R)$ and $\beta \in H^l(X, A; R)$, we have:

$$\alpha \cup \beta = (-1)^{kl} \beta \cup \alpha.$$

Defn. The product map

$$H^*(X;R) \times H^*(Y;R) \xrightarrow{\times} H^*(X\times Y;R)$$

given by $a \times b = p_1^*(a) \times p_2^*(b)$ or is called a cross product or external cup product.

Theorem (Kunneth formula). If X and Y are CW-complexes and $H^k(Y;R)$ is a finitely generated free R-module for all k, then

$$H^*(X;R) \otimes_R H^*(Y;R) \longrightarrow H^*(X\times Y;R)$$

is a ring isom.

Example. $H^*(\mathbb{R}P^\infty \times \mathbb{R}P^\infty; \mathbb{Z}_2)$

$$\cong H^*(\mathbb{R}P^\infty; \mathbb{Z}_2) \otimes H^*(\mathbb{R}P^\infty; \mathbb{Z}_2)$$

$$\cong \mathbb{Z}_2[\alpha] \otimes \mathbb{Z}_2[\beta] \quad \text{(by thm.)}$$

$$\cong \mathbb{Z}_2[\alpha, \beta].$$

Poincaré Duality

Defn. A *manifold of dimension n or an n-manifold* is a Hausdorff space in which each point has a neighborhood homeomorphic to $\mathbb{R}^n$.

A compact manifold is called closed.

Examples. S^n, $\mathbb{RP}^n$, and $\mathbb{CP}^n$ are closed manifolds.

Remark The dimension of a manifold M is intrinsically characterized by the fact that for $x \in M$, $H_i(M, M-\{x\}, \mathbb{Z}) \neq 0$, only for $i=n$. This is because

$$H_i(M, M-\{x\}; \mathbb{Z}) \cong H_i(\mathbb{R}^n, \mathbb{R}^n-\{0\}; \mathbb{Z})$$

(by Excision)

$$\cong \tilde{H}_{i-1}(\mathbb{R}^n-\{0\}; \mathbb{Z})$$

(LES of pair & $\mathbb{R}^n$ is contractible)

$$\cong \tilde{H}_{i-1}(S^{n-1}; \mathbb{Z})$$

$(\mathbb{R}^n-\{0\} \simeq S^{n-1})$.

Defn. A local orientation of M at a point $x \in M$ is a choice of generator μ_x for $H_n(M, M-\{x\})$ which is an infinite cyclic group.

Defn. An orientation of an n-manifold M is a function $x \longmapsto \mu_x$ assigning to each $x \in M$ a local orientation $\mu_x \in H_n(M, M-\{x\})$

satisfying the local consistency condition that each $x \in M$ has a nbhd $U (\approx \mathbb{R}^n)$ containing an open ball $B \ni x$ such that all local orientations μ_y at points $y \in B$ are images of one generator μ_B of $H_n(M, M-B) \cong H_n(\mathbb{R}^n, \mathbb{R}^n - B)$ under the natural maps $H_n(M, M-B) \longrightarrow H_n(M, M-y)$.

<u>Theorem</u>. Every manifold M has a orientable 2-sheeted covering space $\widetilde{M}$.

Proof. As a set, let:

$$\widetilde{M} = \left\{ \mu_x \mid x \in M \text{ and } \mu_x \text{ is a local orientation of } M \text{ at } x \right\}$$

The map $\mu_x \mapsto x$ defines a two-to-on surjection $\widetilde{M} \longrightarrow M$.

To topologize $\widetilde{M}$, given a ball $B \subset M$ (of finite radius) and a generator $\mu_B \in H_n(M, M-B)$, let

$$U(\mu_B) = \left\{ \mu_x \in \widetilde{M} \mid x \in B \text{ and } \mu_x \text{ is the image of } \mu_B \text{ under } H_n(M,M-B) \to H_n(M,M-x) \right\}$$

Then $U(\mu_B)$ forms a basis for a topology on $\widetilde{M}$ and $\widetilde{M} \longrightarrow M$ is a 2-sheeted covering space ∎

Remark. One can imbed $\tilde{M} \to M$ in a larger covering space $M_{\mathbb{Z}} \longrightarrow M$, where:

$$M_{\mathbb{Z}} = \{\alpha_x \in H_n(M, M-x) : x \in M\}$$

As before, we topologize $M_{\mathbb{Z}}$ via the basis $U(\alpha_B) = \{\alpha_x : x \in B\}$ and α_x the image of $\alpha_B \in H_n(M, M-B)$ under $H_n(M, M-B) \longrightarrow H_n(M, M-x)$. Then $M_{\mathbb{Z}} \to M$ is an infinite sheeted cover.

Note that $M_{\mathbb{Z}} = \bigcup_{K=1}^{\infty} M_K$, where $M_0 \approx M$ and:

$$M_K = \{K(\pm\alpha_x) \mid \alpha_x \in H_n(M, M-\{x\}) \text{ and } x \in M\}$$

Defn. A continuous map $M \to M_{\mathbb{Z}}$ of the form $x \mapsto \alpha_x \in H_n(M, M-\{x\})$ is called a *section* of the covering space.

Remark. An orientation is essentially a section $x \mapsto \mu_x$, where $\langle \mu_x \rangle = H_n(M, M-\{x\})$.

Remark. One can generalize the notion of orientation by replacing $\mathbb{Z}$ with R. An *R-orientation* assigns to each $x \in M$, a generator of $H_n(M, M-\{x\}) \cong R$ with the "local condition" where a generator is an element u such that $Ru = R$. ($\Longleftrightarrow$ u is an unit in R since $1 \in R$).

Thus $M_{\mathbb{Z}}$ generalizes to $M_R \longrightarrow M$.

Since $H_n(M, M-\{x\}; R) \approx H_n(M, M-x) \otimes R$

$$M_R = \bigcup_{r \in R} M_r, \text{ where}$$

$$M_r = \{ \pm \mu_x \otimes r \in H_n(M, M-x; R) : x \in M \}.$$

Theorem. Let M be a closed connected n-manifold. Then:

(a) If M is R-orientable, the map $H_n(M;R) \to H_n(M, M-x; R) \approx R$ is an isom $\forall\, x \in M$.

(b) If M is not R-orientable, the map $H_n(M;R) \to H_n(M, M-x; R) \approx R$ is injective with image $\{r \in R \mid 2r = 0\}$ $\forall\, x \in M$.

(c) $H_i(M;R) = 0$ for $i > n$.

In particular,

$$H_n(M;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } M \text{ is orien.} \\ 0, & \text{otherwise.} \end{cases}$$

Defn. An element of $H_n(M;R)$ whose image in $H_n(M, M-x;R)$ is a generator for all x is called a fundamental class for M with coefficients in R.

Corollary. A fundamental class exists iff M is closed and R-orientable.

Proof. ($\Leftarrow$) Follows from earlier theorem.

($\Rightarrow$) Let $\mu \in H_n(M;R)$ be a fund. class and let $\mu_x \in H_n(M, M-x;R)$ be its image. Then $x \longmapsto \mu_x$ is an R-orientation ($\because$ it factors through $H_n(M, M-B;R)$ for any $B \ni x$)

Since, $\mu_x \neq 0$ only for all x in the image in M of a cycle rep μ, which is compact.

Remark. Suppose an n-manifold M has a Δ-complex structure. In simplicial homology a fund. class must be rep. by some linear combination $\sum_i k_i \sigma_i$ of n-simplices σ_i (of M).

Since this maps to a generator of $H_n(M, M-x; \mathbb{Z})$ for all x in interiors (of σ_i), we have: $k_i = \pm 1$ $\forall i$. Also since $\sum k_i \sigma_i$ is a cycle, if σ_i & σ_j share a $(n-1)$-dim^l face, k_i determines k_j.

Thus, it can be seen $\sum k_i \sigma_i$ is a cycle iff M is orientable. With $\mathbb{Z}_2$-coefficients $\sum_i \sigma_i$ is always a cycle.

Corollary. If m is a closed connected n-manifold, then

$$T_{n-1} = \begin{cases} 1 & \text{if } M \text{ is orientable} \\ \mathbb{Z}_2 & \text{if } M \text{ is nonorientable.} \end{cases}$$

Proof. Apply UCT and the fact that homology groups of M are finitely generated.

Duality Theorem

Defn. For an arbitrary space X an a (coefficient) ring R, we define an R-bilinear cap product map:

$$\frown : C_k(X;R) \times C^\ell(X;R) \longrightarrow C_{k-\ell}(X;R)$$

for $k \geq \ell$, by mapping a pair (σ, φ), where $\sigma : \Delta^k \to X$ and $\varphi \in C^\ell(X;R)$ to the singular singular $(k-\ell)$-simplex

$$\sigma \frown \varphi = \varphi(\sigma|[v_0, \dots, v_\ell])\, \sigma|[v_\ell, \dots, v_k]$$

It can be easily verified that:

$$\partial(\sigma \frown \varphi) = (-1)^{\ell}(\partial\sigma \frown \varphi - \sigma \frown \delta\varphi)$$

Thus:

(a) Cap product of a cycle and and cocycle is a cycle.

(b) Cap product of a cycle and coboundary is a boundary.

(c) (a) and (b) $\Rightarrow$ ∃ an induced cap product map:

$$H_K(X;R) \times H^{\ell}(X;R) \xrightarrow{\frown} H_{K-\ell}(X;R).$$

which is R-linear in each variable.

(d) ∃ induced maps:

(i) $H_K(X,A;R) \times H^{\ell}(X;R) \xrightarrow{\frown} H_{K-\ell}(X,A;R)$

(ii) $H_K(X,A;R) \times H^{\ell}(X,A;R) \xrightarrow{\frown} H_{K-\ell}(X;R)$

(iii) $H_K(X,A \cup B;R) \times H^{\ell}(X,A;R) \xrightarrow{\frown} H_{K-\ell}(X,B;R)$, where A and B are open in X.

Lemma. Given a map $f : X \longrightarrow Y$ the relevant induced maps on homology and cohomology fit into the following diagram:

$$\begin{tikzcd}
H_k(X) \times H^\ell(X) \arrow[r, "\frown"] \arrow[d, "f_*"'] & H_{k-\ell}(X) \arrow[d, "f_*"] \\
H_k(Y) \times H^\ell(Y) \arrow[r, "\frown"] \arrow[u, shift left=8, "f^*"'] & H_{k-\ell}(Y)
\end{tikzcd}$$

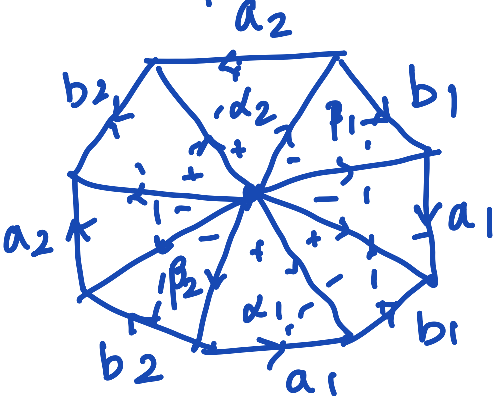
Proof. This follows from the fact that $f_*(\alpha) \frown \varphi = f_*(\alpha \frown f^*(\varphi))$.

Theorem (Poincaré duality). If M is a closed R-orientable M-mfld with fundamental class $[M] \in H_n(M;R)$, then the map

$$D : H^k(M;R) \longrightarrow H_{n-k}(M;R)$$

defined by $D(\alpha) = [M] \frown \alpha$ is an isom. $\forall$ k.

Examples.



A fundamental class $[M]$ generating $H_2(M)$ is represented by the 2-cycle formed by sum of all $4g$ 2-simplices with signs indicated.

Let φ_i (resp. ψ_i) be the cocycle rep α_i (resp. β_i) assigning 1 to a_i (resp b_i) and 0 to others.

Then $[M] \frown \varphi_i = b_i$ and $[M] \frown \psi_i = -a_i$

Thus, b_i is the Poincaré dual to α_i and $-a_i$ is the Poincaré dual of β_i.

In terms of homology, a_i and b_i are Poincaré duals of each other up to sign.

Directed system of groups. Let I be an index set such that for each pair $\alpha, \beta \in I$, $\exists\, \gamma \in I$ such that $\alpha \leq \gamma$ and $\beta \leq \gamma$. Such an index set is called a directed set.

Defn. Let $\{G_\alpha\}_{\alpha \in I}$ be a family of abelian groups indexed with a directed set I. Suppose that:

(a) For each $\alpha, \beta \in I$ with $\alpha \leq \beta$, $\exists$ a hom. $f_{\alpha\beta} : G_\alpha \longrightarrow G_\beta$ such $f_{\alpha\alpha} = 1, \forall \alpha$, and

(b) if $\alpha \leq \beta \leq \gamma$, then

$$f_{\alpha\gamma} = f_{\alpha\beta} \circ f_{\beta\gamma}$$

Then $\{G_\alpha\}_{\alpha \in I}$ is said to form a <u>directed system of groups</u>.

Defn. Given a directed system of groups, the direct limit group $\varinjlim G_\alpha$ is defined as follows:

$$\varinjlim G_\alpha = \bigoplus_{\alpha \in I} G_\alpha \Big/ \langle\langle a - f_{\alpha\beta}(a) : a \in G_\alpha \rangle\rangle,$$

where we view $G_\alpha \subset \bigoplus_\alpha G_\alpha$.

Equivalently,

$$\varinjlim G_\alpha = \bigoplus_{\alpha \in I} G_\alpha \Big/ \sim \, , \quad \text{where}$$

$a \sim b$ if $f_{\alpha\beta}(a) = f_{\beta\gamma}(b)$, for some γ, where $a \in G_\alpha$ and $b \in G_\beta$.

Remark. If $J \subset I$ with the property that for each $\alpha \in I$, $\exists$ $\beta \in J$ with $\alpha \leq \beta$, then $\varinjlim_\alpha G_\alpha$

$$= \varinjlim_{\beta} G_{\beta}$$

In particular, if I has a maximal element γ, then

$$\varinjlim G_{\alpha} = G_{\gamma}.$$

<u>Prop</u>: If a space X is the union of a directed set of subspaces X_{α} with the property that each compact set in X is contained in some X_{α}, then the natural map

$$\varinjlim H_i(X_{\alpha}; G) \longrightarrow H_i(X; G)$$ is an isom. $\forall i$ and G.

<u>Proof</u>. A cycle in X is represented finite sum of singular simplices.

The union of these is compact in X, and hence lies in some X_α. So $\varinjlim H_i(X_\alpha; G) \longrightarrow H_i(X;G)$ is surjective.

If a cycle in some X_α is a boundary in X, compactness would imply it a boundary in some $X_\beta \supset X_\alpha$. $\Rightarrow$ cycle rep. zero in $\varinjlim H_i(X_\alpha; G)$ $\blacksquare$

Cohomology with compact support.

Defn. Let $C_c^i(X;G)$ be the subgroup of $C^i(X;G)$ consisting of cochains $\varphi : C_i(X) \longrightarrow G$ for which $\exists$ a compact set $K = K_\varphi \subset X$ such that φ is zero on all chains in $X-K$.

Then $\delta\varphi$ is zero on chains in $X-K$, so $\delta\varphi \in C_c^{i+1}(X;G)$. Thus the $C_c^i(X;G)$ form a subcomplex of the singular cochain complex of X.

The cohomology groups of $H_c^i(X;G)$ are the cohomology groups with compact supports.

Remark. For a space X, let $\{K_\alpha\}$ be the compact subsets of X. Then $\{K_\alpha\}$ form a directed system under inclusion. For each α, consider $H^i(X, X-K_\alpha; G)$. Then, when $K_\alpha \subset K_\beta$, $\exists$ a natural hom:

$$H^i(X, X-K_\alpha; G) \longrightarrow H^i(X, X-K_\beta; G)$$

Lemma. $H^i_c(X;G) = \varinjlim_{\alpha} H^i(X, X-K_\alpha; G)$

Proof. Let $[z] \in \varinjlim_{\alpha} H^i(X, X-K_\alpha; G)$. Then z is a cocycle in $C^i(X, X-K_\alpha; G)$ for some α. Moreover, z is zero in $\varinjlim H^i(X, X-K_\alpha; G)$ iff

iff $z = \delta y$, for $y \in C^{i-1}(X, X-K_\beta; G)$ for $K_\beta \supset K_\alpha$.

☒

Remark. If X is compact, then $H^i_c(X;G) = H^i(X;G)$.

Example. We wish to compute

$$\varinjlim H^i(\mathbb{R}^n, \mathbb{R}^n - K_\alpha; G)$$

It suffices to consider $\alpha \in \mathbb{Z}^+$ and $K_\alpha = \overline{B(0;\alpha)}$ as every compact set $L \subset \mathbb{R}^n$ is contained in K_α for some α.

Note that

$$H^i(\mathbb{R}^n, \mathbb{R}^n - K_\alpha; G) \cong \begin{cases} G, & \text{if } i=n \\ 0, & \text{otherwise.} \end{cases}$$

Moreover,

$$H^n(\mathbb{R}^n, \mathbb{R}^n - K_\alpha; G) \longrightarrow H^n(\mathbb{R}^n, \mathbb{R}^n - K_{\alpha+1}; G)$$

is an isom.

$$\Longrightarrow H_c^i(\mathbb{R}^n; G) = \begin{cases} 0, & \text{if } i \neq n \\ G, & \text{if } i = n. \end{cases}$$

Homotopy Theory

For a space X with basepoint $x_0 \in X$, define the set $\pi_n(X, x_0)$ to be the homotopy classes of map $f:(I^n, \partial I^n) \longrightarrow (X, x_0)$

For $n \geqslant 2$, consider the operation $+$ on $\pi_n(X, x_0)$ defined by:

$$(f+g)(s_1, \cdots, s_n) = \begin{cases} f(2s_1, s_2, \cdots s_n), & s_1 \in [0, \frac{1}{2}] \\ g(2s_1 - 1, s_2, \cdots s_n), & s_1 \in [\frac{1}{2}, 1] \end{cases}$$

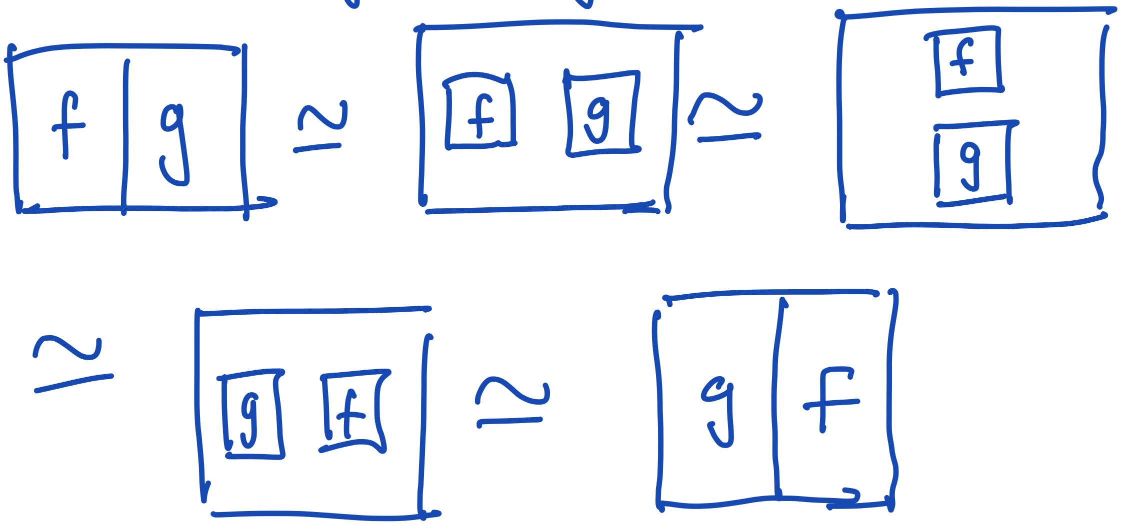
Theorem. For $n \geqslant 1$, $\pi_n(X, x_0)$ is an is a group which is abelian for $n \geqslant 2$.

Proof. The operation is clearly well-defined on the homotopy classes by setting $[f]+[g] := [f+g]$.

We already know that the assertion holds for $n=1$ (as $+$ defines the usual loop concatenation $*$)

Also, since only the first coordinate is involved in $+$ (even for $n \geq 2$), the same arguments as for π_1 show that $\pi_n(X, x_0)$ is a group.

Finally $f+g \simeq g+f$ via the homotopy in the following figures:



Remark.

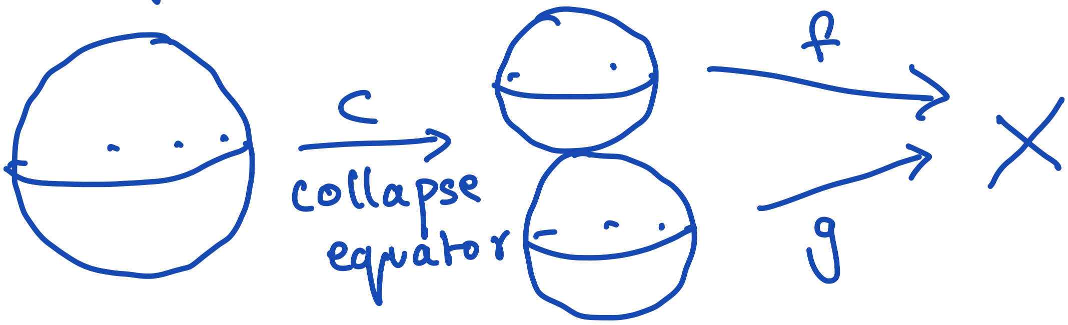
(a) The definition for $\pi_n(X, x_0)$ extends to the case $n=0$ by taking I^0 to be a single point and $\partial I^0 = \phi$. In this case, $\pi_0(X, x_0)$ is not generally a group.

(b) Maps $(I^n, \partial I^n) \longrightarrow (X, x_0)$ are the same as maps

$$\big(I^n/\partial I^n, \underbrace{\partial I^n/\partial I^n}_{s_0}\big) \longrightarrow (X, x_0)$$

ie map $(S^n, s_0) \longrightarrow (X, x_0)$, where homotopies are via maps of the same form.

The operation $f+g$ can be visualized as follows in this case:



Theorem. If X is path-connected different choices of basepoint x_0 produce isomorphic groups $\pi_n(X,x_0)$

Proof. Suppose γ is a path in X from x_0 to x_1

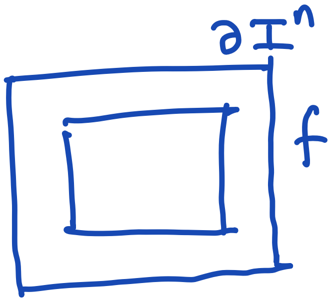
We define an map:

$$\varphi_\gamma : \pi_n(X, x_0) \longrightarrow \pi_n(X, x_1)$$

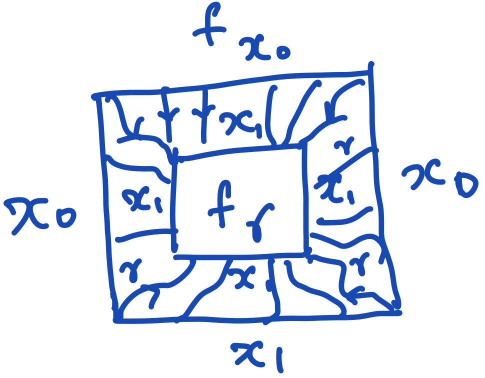
as follows:

$\varphi_\gamma(f) = f_\gamma$, where f_γ is obtained by:

1. Shrinking domain of f into a smaller concentric cube.



2. Inserting path γ on each segment in the shell between the smaller cube and ∂I^n.



3. Set $f_\gamma = \gamma^{-1} \circ f \circ \gamma$

Furthermore, f_γ satisfies the following properties.

(a) $(f+g)_\gamma = f_\gamma + g_\gamma$

(b) $f_{\gamma n} = (fn)_\gamma$

(c) $f_1 = f$

Which (b) & (c) are apparent, (a) can be realized through the homotopy:

$$h_t(s_1,\dots,s_n) = \begin{cases} (f+0)_\gamma((2-t)s_1, s_2, \dots s_n), & s_1 \in [0,\frac{1}{2}] \\ (0+g)((2-t)s_1+t-1, s_2, \dots s_n) & s_1 \in [\frac{1}{2},1] \end{cases}$$

Thus, $(f+g)_\gamma = (f+0)_\gamma + (0+g)_\gamma$

$$= f_\gamma + g_\gamma$$

Consequently, (a) - (c) imply that φ_γ is an isomorphism ▨

Remark $\pi_1(X, x_0)$ acts on $\pi_n(X, x_0)$ via $(\gamma, f) \longmapsto f\gamma$

Since $f\gamma\eta = (f\eta)\gamma$, this induces a hom.

$$\pi_1(X, x_0) \longrightarrow \mathrm{Aut}(\pi_n(X, x_0))$$

When $n=1$, this the action of π_1 on itself by inner automorphisms.

For $n>1$, the action makes $\pi_n(X, x_0)$ into a module over the abelian group ring $\mathbb{Z}[\pi_1(X, x_0)]$.

(Note. $\mathbb{Z}[\pi_1] = \{ \sum n_i \gamma_i : n_i \in \mathbb{Z}, \gamma_i \in \pi_1)$

Thus, the module structure on

π_1 is given by:

$$f_{\sum_i n_i \gamma_i} = \sum_i n_i f_{\gamma_i} \quad \text{for}$$

$f \in \pi_n$.

<u>Remark</u>. π_n is a functor. A continuous map $\varphi : (X, x_0) \longrightarrow (Y, y_0)$ induces a homomorphism

$$\varphi_* : \pi_n(X, x_0) \longrightarrow \pi_n(Y, y_0)$$

defined by $\varphi_*(f) = \varphi \circ f$.

Clearly, φ_* is a well-defined hom.

Prop. A covering space $p:(\tilde{X},x_0)\longrightarrow(X,x_0)$ induces an isomorphism $p_*:\pi_n(\tilde{X},x_0)\longrightarrow\pi_n(X,x_0)$, for $n\geqslant 2$.

Proof. The injectivity follows the same argument as π. Surjectivity follows from the fact for $n\geqslant 2$, every map $(S^n,s_0)\xrightarrow{f}(X,x_0)$ lifts to a map $\tilde{f}:(S^n,s_0)\longrightarrow(\tilde{X},x_0)$ By the lifting crieterion ∎

Corollary. When X has a contractible universal cover, $\pi_n(X,x_0)=0$, for $n\geqslant 2$.

Example

Let $T^n = \prod_{i=1}^{n} S^1$. Then $\pi_i(T^n) = 0$ for $i > 1$.

Proposition. For a product $\prod_\alpha X_\alpha$ of an arbitrary collection of path-connected spaces X_α, $\exists$ an isomorphism:

$$\pi_n\left(\prod_\alpha X_\alpha\right) \cong \prod_\alpha \pi_n(X_\alpha)$$

for all n.

Defn. The relative homotopy groups of a pair (X, A) is defined to be the set of homotopy classes of maps

$$(I^n, \partial I^n, J^n) \longrightarrow (X, A, x_0),$$

where $J^n = \overline{\partial I^n - I^{n-1}}$ and I^{n-1} is the face of I^n obtained by setting the last coordinate as zero.

Remark.

(a) $\pi_0(X, A, x_0)$ is left undefined.

(b) $\pi_n(X, x_0, x_0) = \pi_n(X, x_0)$

(c) $\pi_n(X, A, x_0)$ is a group for $n \geq 2$ under $+$ which is abelian for $n \geq 3$.

(d) For $n=1$, $I^1 = [0,1]$, $I^0 = \{0\}$, and $J^0 = \{1\}$. Thus, $\pi_1(X, A, x_0)$ = Homotopy classes of paths from a varying point in A to a fixed point A. This is not in general a group.

(e) Equivalently, $\pi_n(X, A, x_0)$ = Homotopy classes of map $(D^n, S^{n-1}, s_0) \longrightarrow (X, A, x_0)$.

Lemma (Compression criterion).
A map $f : (D^n, S^{n-1}, s_0) \longrightarrow (X, A, x_0)$ represents zero in $\pi_n(X, A, x_0)$ iff its homotopic rel S^{n-1} to a map whose image is contained

in A.

<u>Proof</u>. <u>($\Longleftarrow$)</u> If such a homotopy exists of f to a map g. Then $[f]=[g]$ in $\pi_n(X,A,x_0)$ and $[g]=0$ via homotopy obtained by composing g with the def ret. of D^n onto s_0.

<u>($\Longrightarrow$)</u> Conversely, let $[f]=0$ via $F: D^n \times I \to X$. Then by restricting F to the family of disks (in $D^n \times I$) starting with $D^n \times \{0\}$ and ending in $D^n \times \{1\} \cup S^{n-1} \times I$, we obtain

a homotopy of f onto a map into A stationary on S^{n-1}.

Theorem. Ther exists an exact sequence

$$\cdots \to \pi_n(A,x_0) \xrightarrow{i_*} \pi_n(X,x_0) \xrightarrow{j_*} \pi_n(X,A,x_0) \xrightarrow{\partial} \pi_{n-1}(A,x_0) \to \cdots \to \pi_0(X,x_0),$$

where i and j are inclusions $(A,x_0) \hookrightarrow (X,x_0)$ and $(X,x_0,x_0) \hookrightarrow (X,A,x_0)$, and ∂ is obtained by restricting $(D^n, S^{n-1}, s_0) \longrightarrow (X,A,x_0)$ to S^{n-1} (or $(I^n, \partial I^n, J^{n-1}) \to (X,A,x_0)$ to I^{n-1}.

∂ is called the boundary map.

Proof

Exactness at $\pi_n(X, B, x_0)$:

$j_* i_* = 0$ as every map $(I^n, \partial I^n, J^{n-1}) \longrightarrow (A, B, x_0)$ represents zero in $\pi_n(X, A, x_0)$ by the compression criterion.

Thus, $\mathrm{Im}(i_*) \subset \mathrm{Ker}(j_*)$

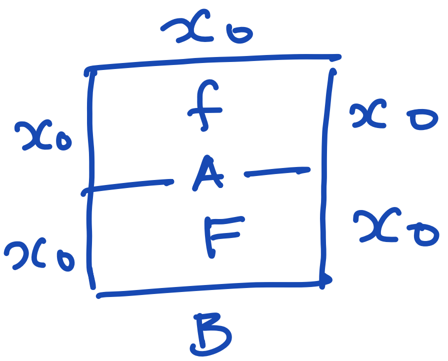
Suppose that $f \in \mathrm{Ker}(j_*)$ i.e. $f: (I^n, \partial I^n, J^{n-1}) \longrightarrow (X, B, x_0)$ represents zero in $\pi_n(X, A, x_0)$. Then by CC, f is homotopic rel ∂I^n to a map with image in A. Hence, $[f] \in \pi_n(X, B, x_0)$ $\in \mathrm{Im}(i_*)$.

Exactness at $\pi_n(X,A,x_0)$.

$\partial j_* = 0$ since the restriction of $(I^n, \partial I^n, J^{n-1}) \to (X,B,x_0)$ to I^{n-1} has image lying in B, and hence represents zero in $\pi_{n-1}(A,B,x_0)$. Thus $\mathrm{Im}(j_*) \subset \mathrm{Ker}(\partial)$.

Conversely let $f \in \mathrm{Ker}(\partial)$ i.e. the restriction of $f:(I^n,\partial I^n,J^{n-1}) \to (X,A,x_0)$ to I^{n-1} represents zero in $\pi_{n-1}(A,B,x_0)$. Then $f|_{I^n} \simeq g$ (with $\mathrm{Im}(g) \subset B$) via $F: I^{n-1} \times I \to A$ (rel ∂I^{n-1}).

We can tack F onto f



to get a map $(I^n, \partial I^n, J^{n-1}) \to (X, B, x_0)$ which is homotopic to f as a map $(I^n, \partial I^{n-1}, J^{n-1}) \to (X, A, x_0)$. So $[f] \in \mathrm{Im}(j_*)$

Exactness at $\pi_n(A, B, x_0)$.

Exercise.

Examples From the LES of the pair (CX, X), we have:

$$\pi_n(CX, X, x_0) \cong \pi_n(X, x_0), \ \forall\, n \geq n$$

In particular, By taking $n=2$ and $X = X_G$ with $\pi_1(X_G) \cong G$, any group G is realized as a relative π_2 group.

Defn. A space (X, x_0) is **n-connected** if $\pi_i(X, x_0) = 0$ for $i \leq n$.

Remark

(a) 0-connected $\Longleftrightarrow$ path-connected
(b) 1-connected $\Longleftrightarrow$ simply-connected

Propostion. The following conditions are equivalent.

(a) Every map $S^i \to X$ is homotopic to a constant map.

(b) Every map $S^i \longrightarrow X$ extends to a map $D^{i+1} \longrightarrow X$

(c) $\pi_i(X, x_0) = 0$ for $x_0 \in X$.

Thus, X is n-connected if any one of (a) – (c) hold for $i \leq n$.

Whitehead Theorem

Theorem. If a map $f: X \to Y$ between CW-complexes induces isomorphisms $f_*: \pi_n(X) \to \pi_n(Y)$ for all n, then f is a homotopy equivalence. In case, f is the inclusion of a subcomplex $X \hookrightarrow Y$, X is a deformation retract of Y.

Defn. A map $f: X \to Y$ between CW-complexes satisfying $f(X^n) \subset Y^n$ $\forall n$ is called a cellular map.

Theorem (Cellular Approximation).

Every map $f: X \longrightarrow Y$ of CW complexes is homotopic to a cellular map. If f is already cellular on a subcomplex $A \subset X$, the homotopy may be taken to be stationary on A.

Corollary. $\pi_n(S^k) = 0$, for $n < k$.

Proof. By CA Theorem, every basepoint-preserving map

$S^n \longrightarrow S^k$ (0-cell taken as basept) can be homotoped relative to basepoint, to be cellular. Hence, it is constant if $n<k$.

<u>Example</u>: $X = \mathbb{RP}^2$, $Y = S^2 \times \mathbb{RP}^\infty$. $\pi_1(X) \cong \pi_1(Y) \cong \mathbb{Z}_2$. Since their universal covers S^2 and $S^2 \times S^\infty$ are homotopically equivalent, it follows that $\pi_n(X) \cong \pi_n(Y)$, for $n \geq 2$.

But $X \not\simeq Y$ since they have non-isomorphic homology groups.